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VAR for VaR: Measuring tail dependence using multivariate regression quantiles[☆]

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ABSTRACT

This paper proposes methods for estimation and inference in multivariate, multi-quantile models. The theory can simultaneously accommodate models with multiple random variables, multiple confidence levels, and multiple lags of the associated quantiles. The proposed framework can be conveniently thought of as a vector autoregressive (VAR) extension to quantile models. We estimate a simple version of the model using market equity returns data to analyze spillovers in the values at risk (VaR) between a market index and financial institutions. We construct impulse-response functions for the quantiles of a sample of 230 financial institutions around the world and study how financial institution-specific and system-wide shocks are absorbed by the system. We show how the long-run risk of the largest and most leveraged financial institutions is very sensitive to market wide shocks in situations of financial distress, suggesting that our methodology can prove a valuable addition to the traditional toolkit of policy makers and supervisors.

1. Introduction

Since the seminal work of Koenker and Bassett (1978), quantile regression models have been increasingly used in many different academic disciplines such as finance, labor economics, and macroeconomics due to their flexibility to allow researchers to investigate the relationship between economic variables not only at the center but also over the entire conditional distribution of the dependent variable. In the early stage, the main development in both theory and application has taken place mainly in the context of cross-section data. However, the application of quantile regression has subsequently moved into the areas of time-series as well as panel data.[2] The whole literature is too vast to be easily summarized, but an excellent and extensive review on many important topics on quantile regression can be found in Koenker (2005).

This paper suggests a multivariate regression quantile model to directly study the degree of tail interdependence among different random variables, therefore contributing to the quantile extension into the time series area in finance. Our theoretical framework allows the quantiles of several random variables to depend on (lagged) quantiles, as well as past innovations and other covariates of interest. This modeling strategy has at least three advantages over the more traditional approaches that rely on the

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1 Hal passed away on 31 March 2012. We miss him dearly and there is no doubt in our mind that the paper would have been much better had he been there helping us with the revision. In particular, he should not be held responsible for any remaining error. Hal was the Chancellor's Associates Distinguished Professor at the Department of Economics, University of California, San Diego.

2 Some relevant and important papers are Koenker and Xiao (2004, 2006), Xiao (2009) in the time-series domain and Abrevaya and Dahl (2008), Lamarche (2010), Galvao (2011) in the panel data setting.

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parameterization of the entire multivariate distribution. First, regression quantile estimates are known to be robust to outliers, a desirable feature in general and for applications to financial data in particular. Second, regression quantile is a semi-parametric technique and as such imposes minimal distributional assumptions on the underlying data generating process (DGP). Third, our multivariate framework allows researchers to directly measure the tail dependence among the random variables of interest, rather than recovering it indirectly via models of time-varying first and second moments.

To illustrate our approach and its usefulness, consider a simple set-up with two random variables, Y_{1t} and Y_{2t}. All information available at time t is represented by the information set $\mathcal{F}_{t-1}$. For a given level of confidence $\theta \in (0, 1)$, the quantile q_{it} at time t for random variables Y_{it} $i = 1, 2$ conditional on $\mathcal{F}_{t-1}$ is

$$\Pr[Y_{it} \leq q_{it}|\mathcal{F}_{t-1}] = \theta, \quad i = 1, 2. \tag{1}$$

A simple version of our proposed structure relates the conditional quantiles of the two random variables according to a vector autoregressive (VAR) structure:

$$q_{1t} = X_t'\beta_1 + b_{11}q_{1t-1} + b_{12}q_{2t-1},$$
$$q_{2t} = X_t'\beta_2 + b_{21}q_{1t-1} + b_{22}q_{2t-1},$$

where X_t represents predictors belonging to $\mathcal{F}_{t-1}$ and typically includes lagged values of Y_{it}. If $b_{12} = b_{21} = 0$, the above model reduces to the univariate CAViaR model of Engle and Manganelli (2004), and the two specifications can be estimated independently from each other. If, however, b_{12} and/or b_{21} are different from zero, the model requires the joint estimation of all of the parameters in the system. The off-diagonal coefficients b_{12} and b_{21} represent the measure of tail codependence between the two random variables, thus the hypothesis of no tail codependence can be assessed by testing $H_0 : b_{12} = b_{21} = 0$.

The first part of this paper develops the consistency and asymptotic theory for the multivariate regression quantile model. Our fully general model is much richer than the above example, as we can accommodate: (i) more than two random variables; (ii) multiple lags of q_{it}; and (iii) multiple confidence levels, say $(\theta_1, \ldots, \theta_p)$.

In the second part of this paper, as an empirical illustration of the model, we estimate a series of bivariate VAR models for the conditional quantiles of the return distributions of individual financial institutions from around the world. Since quantiles represent one of the key inputs for the computation of the Value at Risk (VaR)[3] for financial assets, we call this model VAR for VaR, that is, a vector autoregressive (VAR) model where the dependent variables are the VaR of the financial institutions, which are dependent on (lagged) VaR and past shocks.

Our modeling framework appears particularly suitable to develop sound measures of financial spillover, the importance of which has been brought to the forefront by the recent financial crisis. In the current policy debate, great emphasis has been put on how to measure the additional capital needed by financial institutions in a situation of generalized market distress. The logic is that if the negative externality associated with the spillover of risks within the system is not properly internalized, banks may find themselves in need of additional capital at exactly the worst time, such as when it is most difficult and expensive to raise fresh new capital. If the stability of the whole system is threatened, taxpayer money has to be used to backstop the financial system, to avoid systemic bank failures that may bring the whole economic system to a collapse.[4]

Adrian and Brunnermeier (2009) and Acharya et al. (2010) have recently proposed to classify financial institutions according to the sensitivity of their VaR to shocks to the whole financial system. The empirical section of this paper illustrates how the multivariate regression quantile model provides an ideal framework to estimate directly the sensitivity of VaR of a given financial institution to system-wide shocks. A useful by-product of our modeling strategy is the ability to compute quantile impulse-response functions. Using the quantile impulse-response functions, we can assess the resilience of financial institutions to shocks to the overall index, as well as their persistence.

The model is estimated on a sample of 230 financial institutions from around the world. For each of these equity return series, we estimate a bivariate VAR for VaR where one variable is the return on a portfolio of financial institutions and the other variable is the return on the single financial institution. We find strong evidence of significant tail codependence for a large fraction of the financial institutions in our sample. When aggregating the impulse response functions at the sectorial and geographic level no striking differences are revealed. We, however, find significant cross-sectional differences. By aggregating the 30 stocks with the largest and smallest market value (thus, forming two portfolios), we find that, in tranquil times, the two portfolios have comparable risk. In times of severe financial distress, however, the risk of the first portfolio increases disproportionately relative to the second. Similar conclusions are obtained when aggregation is done according to the most and least leveraged institutions. These results hold for both in-sample and out-of-sample.

The plan of this paper is as follows. In Section 2, we set forth the multivariate and multi-quantile CAViaR framework, a generalization of Engle and Manganelli's original CAViaR 2004 model. Section 3 provides conditions ensuring the consistency and asymptotic normality of the estimator, as well as the results which provide a consistent asymptotic covariance matrix estimator. Section 4 contains an example of a data generating process which is consistent with the proposed multivariate quantile model, while Section 5 introduces the long run quantile impulse-response functions and derives the associated standard errors. Section 6 contains the empirical application. Section 7 provides a summary and some concluding remarks. Appendix contains all of the technical proofs of the theorems in the text.

2. The multivariate and multi-quantile process and its model

We consider a sequence of random variables denoted by $\{(Y_t', X_t') \ : \ t = 1, 2, \ldots, T\}$ where Y_t is a finitely dimensioned $n \times 1$ vector and X_t is also a countably dimensioned vector whose first element is one. To fix ideas, Y_t can be considered as the dependent variables and X_t as the explanatory variables in a typical regression framework. In this sense, the proposed model which will be developed below is sufficiently general enough to handle multiple dependent variables. We specify the data generating process as follows.

Assumption 1. The sequence $\{(Y_t', X_t')\}$ is a stationary and ergodic stochastic process on the complete probability space $(\Omega, \mathcal{F}, P_0)$,

3 An extensive discussion on how to properly use quantile regression to estimate VaR can be found in Chernozhukov and Umantsev (2001) in which they also emphasize the importance of using extremal or near-extremal quantile regression.

4 It should be emphasized that the proposed method measures the degree of tail dependence between variables in a predictive manner, as in a GARCH framework. Since the tail risk metric of a given variable is affected only by lagged or past tail-risk metrics of other variables, the contemporaneous tail dependence cannot be measured in our framework.

where Ω is the sample space, $\mathcal{F}$ is a suitably chosen σ-field, and P_0 is the probability measure providing a complete description of the stochastic behavior of the sequence of $\{(Y_t', X_t')\}$.

We define $\mathcal{F}_{t-1}$ to be the σ-algebra generated by $Z^{t-1} := \{X_t, (Y_{t-1}, X_{t-1}), (Y_{t-2}, X_{t-2}), \ldots\}$, i.e. $\mathcal{F}_{t-1} := \sigma(Z^{t-1})$. For $i = 1, \ldots, n$, we also define $F_{it}(y) := P_0[Y_{it} < y \mid \mathcal{F}_{t-1}]$ which is the cumulative distribution function (CDF) of Y_{it} conditional on $\mathcal{F}_{t-1}$. In the quantile regression literature, it is typical to focus on a specific quantile index; for example, $\theta \in (0, 1)$. In this paper, we will develop a more general quantile model where multiple quantile indexes can be accounted for jointly. To be more specific, we consider p quantile indexes denoted by $\theta_{i1}, \theta_{i2}, \ldots, \theta_{ip}$ for the ith element (denoted by Y_{it}) of Y_t. The p quantile indexes do not need to be the same for all of the elements of Y_t, which explains the double indexing of θ_{ij}. Moreover, we note that we specify the same number (p) of quantile indexes for each $i = 1, \ldots, n$. However, this is just for notational simplicity. Our theory easily applies to the case in which the number of quantile indexes differs with i; i.e., p can be replaced with p_i.

To formalize our argument, we assume that the quantile indexes are ordered such that $0 < \theta_{i1} < \cdots < \theta_{ip} < 1$. For $j = 1, \ldots, p$, the $\theta_{ij}th$-quantile of Y_{it} conditional on $\mathcal{F}_{t-1}$, denoted $q^*_{i,j,t}$, is

$$q^*_{i,j,t} := \inf\{y : F_{it}(y) \geq \theta_{ij}\}, \tag{2}$$

and if F_{it} is strictly increasing,

$$q^*_{i,j,t} = F_{it}^{-1}(\theta_{ij}).$$

Alternatively, $q^*_{i,j,t}$ can be represented as

$$\int_{-\infty}^{q^*_{i,j,t}} dF_{it}(y) = E[1_{[Y_{it} \leq q^*_{i,j,t}]} \mid \mathcal{F}_{t-1}] = \theta_{ij}, \tag{3}$$

where $dF_{it}(\cdot)$ is the Lebesgue–Stieltjes probability density function (PDF) of Y_{it} conditional on $\mathcal{F}_{t-1}$, corresponding to F_{it}.

Our objective is to jointly estimate the conditional quantile functions $q^*_{i,j,t}$ for $i = 1, \ldots, n$ and $j = 1, 2, \ldots, p$. For this, we write $q^*_t := (q^{*\prime}_{1,t}, q^{*\prime}_{2,t}, \ldots, q^{*\prime}_{n,t})'$ with $q^*_{i,t} := (q^*_{i,1,t}, q^*_{i,2,t}, \ldots, q^*_{i,p,t})'$ and impose an additional appropriate structure. First, we ensure that the conditional distributions of Y_{it} are everywhere continuous, with positive densities at each of the conditional quantiles of interest, $q^*_{i,j,t}$. We let f_{it} denote the conditional probability density function (PDF) which corresponds to F_{it}. In stating our next condition (and where helpful elsewhere), we make explicit the dependence of the conditional CDF F_{it} on $\omega \in \Omega$ by writing $F_{it}(\omega, y)$ in place of $F_{it}(y)$. Similarly, we may write $f_{i,t}(\omega, y)$ in place of $f_{i,t}(y)$. The realized values of the conditional quantiles are correspondingly denoted as $q^*_{i,j,t}(\omega)$.

Our next assumption ensures the desired continuity and imposes specific structure on the quantiles of interest.

Assumption 2. (i) Y_{it} is continuously distributed such that for each $\omega \in \Omega$, $F_{it}(\omega, \cdot)$ and $f_{it}(\omega, \cdot)$ are continuous on $\mathbb{R}$, $t = 1, 2, \ldots, T$; (ii) For the given $0 < \theta_{i1} < \cdots < \theta_{ip} < 1$ and $\{q^*_{i,j,t}\}$ as defined above, we suppose the following: (a) for each i, j, t, and ω, $f_{it}(\omega, q^*_{i,j,t}(\omega)) > 0$; and (b) for the given finite integers k and m, there exist a stationary ergodic sequence of random $k \times 1$ vectors $\{\Psi_t\}$, with Ψ_t measurable-$\mathcal{F}_{t-1}$, and real vectors $\beta^*_{ij} := (\beta^*_{i,j,1}, \ldots, \beta^*_{i,j,k})'$ and $\gamma^*_{i,j,\tau} := (\gamma^{*\prime}_{i,j,\tau,1}, \ldots, \gamma^{*\prime}_{i,j,\tau,n})'$, where each $\gamma^*_{i,j,\tau,k}$ is a $p \times 1$ vector, such that for $i = 1, \ldots, n$, $j = 1, \ldots, p$, and all t,

$$q^*_{i,j,t} = \Psi_t' \beta^*_{ij} + \sum_{\tau=1}^{m} q^{*\prime}_{t-\tau} \gamma^*_{i,j,\tau}. \tag{4}$$

The structure of equation in (4) is a multivariate version of the MQ-CAViaR process of White et al. (2008), itself a multi-quantile version of the CAViaR process introduced by Engle and Manganelli (2004). Under suitable restrictions on $\gamma^*_{i,j,\tau}$, we obtain as special cases; (1) separate MQ-CAViaR processes for each element of Y_t; (2) standard (single quantile) CAViaR processes for each element of Y_t; or (3) multivariate CAViaR processes, in which a single quantile of each element of Y_t is related dynamically to the single quantiles of the (lags of) other elements of Y_t. Thus, we call any process that satisfies our structure "Multivariate MQ-CAViaR" (MVMQ-CAViaR) processes or naively "VAR for VaR".

For MVMQ-CAViaR, the number of relevant lags can differ across the elements of Y_t and the conditional quantiles; this is reflected in the possibility that for the given j, elements of $\gamma^*_{i,j,\tau}$ may be zero for values of τ greater than some given integer. For notational simplicity, we do not represent m as being dependent on i or j. Nevertheless, by convention, for no $\tau \leq m$ does $\gamma^*_{i,j,\tau}$ equal the zero vector for all i and j. The finitely dimensioned random vectors Ψ_t may contain lagged values of Y_t, as well as measurable functions of X_t and lagged X_t. In particular, Ψ_t may contain Stinchcombe and White's (1998) GCR transformations, as discussed in White (2006).

For a particular quantile, say θ_{ij}, the coefficients to be estimated are β^*_{ij} and $\gamma^*_{ij} := (\gamma^{*\prime}_{i,j,1}, \ldots, \gamma^{*\prime}_{i,j,m})'$. Let $\alpha^{*\prime}_{ij} := (\beta^{*\prime}_{ij}, \gamma^{*\prime}_{ij})$, and write $\alpha^* = (\alpha^{*\prime}_{11}, \ldots, \alpha^{*\prime}_{1p}, \ldots, \alpha^{*\prime}_{n1}, \ldots, \alpha^{*\prime}_{np})'$, an $\ell \times 1$ vector, where $\ell := np(k + npm)$. We call α^* the "MVMQ-CAViaR coefficient vector". We estimate α^* using a correctly specified model for the MVMQ-CAViaR process. First, we specify our model in the following assumption.

Assumption 3. (i) Let $\mathbb{A}$ be a compact subset of $\mathbb{R}^\ell$. For $i = 1, \ldots, n$, and $j = 1, \ldots, p$, we suppose the following: (a) the sequence of functions $\{q_{i,j,t} : \Omega \times \mathbb{A} \to \mathbb{R}^{p_i}\}$ is such that for each t and each $\alpha \in \mathbb{A}$, $q_{i,j,t}(\cdot, \alpha)$ is measurable-$\mathcal{F}_{t-1}$; (b) for each t and each $\omega \in \Omega$, $q_{i,j,t}(\omega, \cdot)$ is continuous on $\mathbb{A}$; and (c) for each i, j, and t, $q_{i,j,t}(\cdot, \alpha)$ is specified as follows:

$$q_{i,j,t}(\cdot, \alpha) = \Psi_t' \beta_{ij} + \sum_{\tau=1}^{m} q_{t-\tau}(\cdot, \alpha)' \gamma_{i,j,\tau}. \tag{5}$$

Next, we impose the correct specification assumption together with an identification condition. Assumption 4(i.a) delivers the correct specification by ensuring that the MVMQ-CAViaR coefficient vector α^* belongs to the parameter space, $\mathbb{A}$. This ensures that α^* optimizes the estimation objective function asymptotically. Assumption 4(i.b) delivers the identification by ensuring that α^* is the only optimizer. In stating the identification condition, we define $\delta_{i,j,t}(\alpha, \alpha^*) := q_{i,j,t}(\cdot, \alpha) - q_{i,j,t}(\cdot, \alpha^*)$ and use the norm $\|\alpha\| := \max_{s=1,\ldots,\ell} |\alpha_s|$, where for convenience we also write $\alpha = (\alpha_1, \ldots, \alpha_\ell)'$.

Assumption 4. (i) (a) There exists $\alpha^* \in \mathbb{A}$ such that for all i, j, t,

$$q_{i,j,t}(\cdot, \alpha^*) = q^*_{i,j,t}; \tag{6}$$

(b) There is a non-empty index set $\mathcal{I} \subseteq \{(1, 1), \ldots, (1, p), \ldots, (n, 1), \ldots, (n, p)\}$ such that for each $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that for all $\alpha \in \mathbb{A}$ with $\|\alpha - \alpha^*\| > \epsilon$,

$$P[\cup_{(i,j) \in \mathcal{I}} \{|\delta_{i,j,t}(\alpha, \alpha^*)| > \delta_\epsilon\}] > 0.$$

Among other things, this identification condition ensures that there is sufficient variation in the shape of the conditional distribution to support the estimation of a sufficient number ($\#\mathcal{I}$) of the variation-free conditional quantiles. As in the case of MQ-CAViaR, distributions that depend on a given finite number of variation-free parameters, say r, will generally be able to support r

variation-free quantiles. For example, the quantiles of the $N(\mu, 1)$ distribution all depend on μ alone, so there is only one "degree of freedom" for the quantile variation. Similarly, the quantiles of the scaled and shifted t-distributions depend on three parameters (location, scale, and kurtosis), so there are only three "degrees of freedom" for the quantile variation.

3. Asymptotic theory

We estimate α^* by the quasi-maximum likelihood method. Specifically, we construct a quasi-maximum likelihood estimator (QMLE) $\hat{\alpha}_T$ as the solution to the optimization problem

$$\min_{\alpha \in \mathbb{A}} \bar{S}_T(\alpha) := T^{-1} \sum_{t=1}^{T} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{p} \rho_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot, \alpha)) \right\}, \tag{7}$$

where $\rho_\theta(e) = e\psi_\theta(e)$ is the standard "check function," defined using the usual quantile step function, $\psi_\theta(e) = \theta - 1_{[e \le 0]}$.

We thus view

$$S_t(\alpha) := -\left\{ \sum_{i=1}^{n} \sum_{j=1}^{p} \rho_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot, \alpha)) \right\}$$

as the quasi log-likelihood for the observation t. In particular, $S_t(\alpha)$ is the log-likelihood of a vector of np independent asymmetric double exponential random variables (see White, 1994, Chapter 5.3; Kim and White, 2003 and Komunjer, 2005). Because $Y_{it} - q_{i,j,t}(\cdot, \alpha)$ does not need to actually have this distribution, the method can be regarded as a *quasi* maximum likelihood.

Once the QML estimator $\hat{\alpha}_T$ is obtained, one can compute the estimated conditional quantile functions $\hat{q}_{i,j,t} = q_{i,j,t}(\hat{\alpha}_T)$. Considering the natural monotonicity property of quantile functions, it is expected that $\hat{q}_{i,1,t} \le \hat{q}_{i,2,t} \le \cdots \le \hat{q}_{i,p,t}$ because $\theta_{i1} < \theta_{i2} < \cdots < \theta_{ip}$. However, when multiple quantiles are jointly estimated, such a desirable ordering can be sometimes violated; that is, some estimated quantile functions can cross each other, which is known as the 'quantile crossing' problem. If the quantile model in (5) is correctly specified as imposed in Assumption 4(i), then the population quantile functions are monotonic and the estimated quantile functions will converge to the corresponding population quantile functions. Hence, the quantile crossing problem is simply a finite sample problem in such a case, and should be negligible when the sample size is sufficiently large. If either the quantile model is misspecified or the sample size is not large enough, then the quantile crossing problem can still be of concern. In that case, one can use some recently developed techniques to correct the problem such as the monotonization method by Chernozhukov et al. (2010) or the isotonization method suggested by Mammen (1991).[5] In passing, we note that in the subsequent empirical study later, we exclusively focus on estimating the MVMQ-CAViaR model at the 1% level only (i.e. $p = 1$ and $\theta = 0.01$) so that there is no quantile crossing problem in our example.

We establish consistency and asymptotic normality for $\hat{\alpha}_T$ through methods analogous to those of White et al. (2008). For conciseness, we place the remaining regularity conditions (i.e., Assumptions 5–7) and technical discussions in the Appendix.

[5] Since the former is known to outperform the latter in quantile regression models, we briefly explain the monotonization method only. Given the estimated quantile function $q_{i,j,t}(\hat{\alpha}_T)$, we can define a random variable $Y_{\mathcal{F}} = q_{i,\theta_j = U,t}(\hat{\alpha}_T)$ where U is the standard uniform random variable over the unit interval [0, 1]. The θth-quantile of $Y_{\mathcal{F}}$ denoted by $q^m_{i,j,t}(\hat{\alpha}_T)$ is monotone with respect to θ_j by construction. Hence, it is taken as a monotonized version of the original estimated quantile function $q_{i,j,t}(\hat{\alpha}_T)$.

Theorem 1. *Suppose that Assumptions 1, 2(i, ii), 3(i), 4 (i) and 5(i, ii) hold. Then, we have*

$$\hat{\alpha}_T \xrightarrow{a.s.} \alpha^*.$$

Next we will show that $\hat{\alpha}_T$ is asymptotically normal. For this, we define the "error" $\varepsilon_{i,j,t} := Y_{it} - q_{i,j,t}(\cdot, \alpha^*)$ and let $f_{i,j,t}(\cdot)$ be the density of $\varepsilon_{i,j,t}$ conditional on $\mathcal{F}_{t-1}$. We also define $\nabla q_{i,j,t}(\cdot, \alpha)$ as the $\ell \times 1$ gradient vector of $q_{i,j,t}(\cdot, \alpha)$ differentiated with respect to α. With Q^* and V^* as given below, the asymptotic normality result is provided in the following theorem.

Theorem 2. *Suppose that Assumptions 1–6 hold. Then, the asymptotic distribution of the QML estimator $\hat{\alpha}_T$ obtain from (7) is given by:*

$$T^{1/2}(\hat{\alpha}_T - \alpha^*) \xrightarrow{d} N(0, Q^{*-1}V^*Q^{*-1}),$$

where

$$Q^* := \sum_{i=1}^{n} \sum_{j=1}^{p} E[f_{i,j,t}(0) \nabla q_{i,j,t}(\cdot, \alpha^*) \nabla' q_{i,j,t}(\cdot, \alpha^*)],$$

$$V^* := E(\eta_t^* \eta_t^{*\prime}),$$

$$\eta_t^* := \sum_{i=1}^{n} \sum_{j=1}^{p} \nabla q_{i,j,t}(\cdot, \alpha^*) \psi_{\theta_{ij}}(\varepsilon_{i,j,t}),$$

$$\varepsilon_{i,j,t} := Y_{it} - q_{i,j,t}(\cdot, \alpha^*).$$

We note that the transformed error term of $\psi_{\theta_{ij}}(\varepsilon_{i,j,t}) = \theta_{ij} - 1_{[\varepsilon_{i,j,t} \le 0]}$ appearing in Theorem 2 can be viewed as a generalized residual. Theorem 2 shows that the asymptotic behavior of the QML estimator $\hat{\alpha}_T$ is well described by the usual normal law. We emphasize that one particular condition that has implicitly played an important role for obtaining such a usual normal law is that all of quantile indexes $\theta_{i1}, \theta_{i2}, \ldots, \theta_{ip}$ are fixed as $T \to \infty$. There have been important developments (see Chernozhukov, 2005, and Chernozhukov and Fernandez-Val, 2011) based on the extreme value (EV) theory in statistics about the asymptotic behavior of θth regression quantiles under the condition that the quantile index θ converges to zero as $T \to \infty$, which is referred to as 'extremal quantile regression'. This approach intends to provide a better approximation (called the EV asymptotic law) to the finite sample distribution of the θth quantile estimator than the usual normal law when the quantile index θ is fairly small relative to the sample size. It might be interesting to apply the extremal quantile regression method to our setting, but it is beyond the scope of the current paper. Hence, we will assume that all of quantile indexes $\theta_{i1}, \theta_{i2}, \ldots, \theta_{ip}$ are fixed as $T \to \infty$ for the rest of the paper.

To test restrictions on α^* or to obtain confidence intervals, we require a consistent estimator of the asymptotic covariance matrix $C^* := Q^{*-1}V^*Q^{*-1}$. First, we provide a consistent estimator $\hat{V}_T$ for V^*; then we propose a consistent estimator $\hat{Q}_T$ for Q^*. Once $\hat{V}_T$ and $\hat{Q}_T$ are proved to be consistent for V^* and Q^* respectively, then it follows by the continuous mapping theorem that $\hat{C}_T := \hat{Q}_T^{-1}\hat{V}_T\hat{Q}_T^{-1}$ is a consistent estimator for C^*.

A straightforward plug-in estimator of V^* is constructed as follows:

$$\hat{V}_T := T^{-1} \sum_{t=1}^{T} \hat{\eta}_t \hat{\eta}_t',$$

$$\hat{\eta}_t := \sum_{i=1}^{n} \sum_{j=1}^{p} \nabla q_{i,j,t}(\cdot, \hat{\alpha}_T) \psi_{\theta_{ij}}(\hat{\varepsilon}_{i,j,t}),$$

$$\hat{\varepsilon}_{i,j,t} := Y_{it} - q_{i,j,t}(\cdot, \hat{\alpha}_T).$$

The next result establishes the consistency of $\hat{V}_T$ for V^*.

Theorem 3. *Suppose that Assumptions 1–6 hold. Then, we have the following result:*

$$\hat{V}_T \xrightarrow{p} V^*.$$

Next, we provide a consistent estimator of Q^*. We follow Powell's (1984) suggestion of estimating $f_{i,j,t}(0)$ with $1_{[-\hat{c}_T \le \hat{\varepsilon}_{i,j,t} \le \hat{c}_T]}/2\hat{c}_T$ for a suitably chosen sequence $\{\hat{c}_T\}$. This is also the approach taken in Kim and White (2003), Engle and Manganelli (2004), and White et al. (2008). Accordingly, our proposed estimator is

$$\hat{Q}_T = (2\hat{c}_T T)^{-1} \sum_{i=1}^{n} \sum_{t=1}^{T} \sum_{j=1}^{p} 1_{[-\hat{c}_T \le \hat{\varepsilon}_{i,j,t} \le \hat{c}_T]} \nabla q_{i,j,t}(\cdot, \hat{\alpha}_T) \times \nabla' q_{i,j,t}(\cdot, \hat{\alpha}_T).$$

Theorem 4. *Suppose that Assumptions 1–7 hold. Then, we obtain the consistency result for $\hat{Q}_T$ as follows:*

$$\hat{Q}_T \xrightarrow{p} Q^*.$$

There is no guarantee that $\hat{\alpha}_T$ is asymptotically efficient. There is now considerable literature that investigates the efficient estimation in quantile models; see, for example, Newey and Powell (1990), Otsu (2003) and Komunjer and Vuong (2006, 2007a,b). Thus far, this literature has only considered single quantile models. It is not obvious how the results for the single quantile models extend to multivariate and multi-quantile models. Nevertheless, Komunjer and Vuong (2007a) show that the class of QML estimators is not large enough to include an efficient estimator, and that the class of M-estimators, which strictly includes the QMLE class, yields an estimator that attains the efficiency bound. Specifically, when $p = n = 1$, they show that replacing the usual quantile check function $\rho_{\theta_{ij}}(\cdot)$ in Eq. (7) with

$$\rho^*_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot, \alpha)) = (\theta_{ij} - 1_{[Y_{it} - q_{i,j,t}(\cdot,\alpha) \le 0]})(F_{it}(Y_{it}) - F_{it}(q_{i,j,t}(\cdot, \alpha)))$$

will deliver an asymptotically efficient quantile estimator. We conjecture that replacing $\rho_{\theta_{ij}}$ with $\rho^*_{\theta_{ij}}$ in equation in (7) will improve the estimator efficiency for p and/or n greater than 1. Another promising efficiency improvement is the application of the semiparametric SUR-type quantile estimator proposed by Jun and Pinkse (2009) for multiple quantile equations. Our method implicitly assumes that the generalized errors $\psi_{\theta_{ij}}(\varepsilon_{i,j,t}) = \theta_{ij} - 1_{[\varepsilon_{i,j,t} \le 0]}$ appearing in Theorem 2 are uncorrelated between different equations and different quantiles. This assumption is rather strict, and the estimation procedure in Jun and Pinkse (2009) is designed to improve efficiency when these errors are correlated in linear quantile models. As such, additional work may be required to make the procedure applicable in the context of non-linear quantile models as in our framework. This is an interesting topic for future work.

4. An example of a data generating process

In this section, we provide an example of a data generating process that can generate the MVMQ-CAViaR model analyzed in the previous sections. To fix ideas, we consider a situation where we observe two random variables (Y_{1t} and Y_{2t}). For instance, the first one Y_{1t} could represent the per-period return on a large portfolio or a financial index consisting of sufficiently many financial institutions, while the second Y_{2t} is the per-period return on a specific financial institution within the portfolio or the index. A possible data generating process for $Y_t = (Y_{1t}, Y_{2t})'$ can be specified as follows:

$$\begin{bmatrix} Y_{1t} \\ Y_{2t} \end{bmatrix} = \begin{bmatrix} \alpha_t & 0 \\ \beta_t & \gamma_t \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}, \tag{8}$$

where α_t, β_t and γ_t are F_t-measurable, and each element of $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ has the standard normal distribution and is mutually independent and identically distributed (IID). The triangular structure in (8) reflects the plausible restriction that shocks to the large portfolio are allowed to have a direct impact on the return of the specific asset, but shocks to the specific asset do not have a direct impact on the whole portfolio.

We note that the standard deviations of Y_{1t} and Y_{2t} are given by $\sigma_{1t} = \alpha_t$ and $\sigma_{2t} = \sqrt{\beta_t^2 + \gamma_t^2}$ respectively. Further, let α_t, β_t and γ_t be specified to satisfy the following usual GARCH-type restrictions:

$$\begin{aligned} \sigma_{1t} &= \tilde{c}_1 + \tilde{a}_{11}|Y_{1t-1}| + \tilde{a}_{12}|Y_{2t-1}| + \tilde{b}_{11}\sigma_{1t-1} + \tilde{b}_{12}\sigma_{2t-1}, \\ \sigma_{2t} &= \tilde{c}_2 + \tilde{a}_{21}|Y_{1t-1}| + \tilde{a}_{22}|Y_{2t-1}| + \tilde{b}_{21}\sigma_{1t-1} + \tilde{b}_{22}\sigma_{2t-1}. \end{aligned} \tag{9}$$

We note that $q_{it} = \sigma_{it}\Phi^{-1}(\theta)$, $i = \{1, 2\}$ where $\Phi(z)$ is the cumulative distribution function of $N(0, 1)$. Hence, by substituting the result $\sigma_{it} = \Phi(\theta)q_{it}$ in (9), it can be formally shown that the respective θth-quantile processes associated with this DGP are given by the following form denoted as 'MVMQ-CAViaR(1, 1)':

$$\begin{aligned} q_{1t} &= c_1(\theta) + a_{11}(\theta)|Y_{1t-1}| + a_{12}(\theta)|Y_{2t-1}| \\ &\quad + b_{11}(\theta)q_{1t-1} + b_{12}(\theta)q_{2t-1}, \\ q_{2t} &= c_2(\theta) + a_{21}(\theta)|Y_{1t-1}| + a_{22}(\theta)|Y_{2t-1}| \\ &\quad + b_{21}(\theta)q_{1t-1} + b_{22}(\theta)q_{2t-1}, \end{aligned} \tag{10}$$

where $c_i(\theta) = \tilde{c}_i\Phi^{-1}(\theta)$, $a_{ij}(\theta) = \tilde{a}_{ij}\Phi^{-1}(\theta)$, $b_{ij}(\theta) = \tilde{b}_{ij}$. The bivariate quantile model in (10) can be written more compactly in matrix form as follows:

$$q_t = c + A|Y_{t-1}| + Bq_{t-1}, \tag{11}$$

where q_t, Y_{t-1}, and c are 2-dimensional vectors, and A, B are (2, 2)-matrices whose elements are obviously shown in (10).

5. The pseudo quantile impulse response function

In this section, we discuss how an impulse response function can be developed in the proposed MVMQ-CAViaR framework. For this, we assume that the conditional quantiles of Y_t follow the simple MVMQ-CAViaR(1,1) model in (11). Since the DGP is not fully specified in quantile regression models, it is not obvious how to derive impulse response functions from structural shocks. Unlike the standard impulse response analysis where a one-off intervention δ is given to the error term ε_t, we will assume that the one-off intervention δ is given to the observable Y_{1t} only at time t so that $\tilde{Y}_{1t} := Y_{1t} + \delta$. In all other times there is no change in Y_{1t}. In other words, the time path of Y_{1t} without the intervention would be

$$\{\ldots, Y_{1t-2}, Y_{1t-1}, Y_{1t}, Y_{1t+1}, Y_{1t+2}, \ldots\}$$

while the time path with the intervention would be

$$\{\ldots, Y_{1t-2}, Y_{1t-1}, \tilde{Y}_{1t}, Y_{1t+1}, Y_{1t+2}, \ldots\}.$$

We acknowledge that the set-up is fairly restrictive because it ignores the dynamic evolution in the second moment of Y_{1t} specified by (9), which forces no change in Y_{1t+s} for $s \ge 1$. However, this seems to be the only plausible way to obtain an impulse response function under the conditional quantile model that we consider, and such a strong limitation should be borne in mind when we discuss the empirical results in Section 6. To distinguish our approach from the standard one, the derived function tracing the effect of the one-off impulse δ given to Y_{1t} will be called the pseudo impulse response function.[6]

Our objective is to measure the impact of the one-off intervention at time t on the quantile dynamics. The pseudo θth-quantile impulse-response function (QIRF) for the ith variable (Y_{it}) denoted as $\Delta_{i,s}(\tilde{Y}_{1t})$ is defined as

$$\Delta_{i,s}(\tilde{Y}_{1t}) = \tilde{q}_{i,t+s} - q_{i,t+s}, \quad s = 1, 2, 3, \ldots$$

where $\tilde{q}_{i,t+s}$ is the θth-conditional quantile of the affected series ($\tilde{Y}_{it+s}$) and $q_{i,t+s}$ is the θth-conditional quantile of the unaffected series (Y_{it+s}).

First, we consider the case for $i = 1$, i.e. $\Delta_{1,s}(\tilde{Y}_{1t})$. When $s = 1$, the pseudo QIRF is given by

$$\Delta_{1,1}(\tilde{Y}_{1t}) = a_{11}(|\tilde{Y}_{1t}| - |Y_{1t}|) + a_{12}(|\tilde{Y}_{2t}| - |Y_{2t}|).$$

For $s > 1$, the pseudo QIRF is given by

$$\Delta_{1,s}(\tilde{Y}_{1t}) = b_{11}\Delta_{1,s-1}(\tilde{Y}_{1t}) + b_{12}\Delta_{2,s-1}(\tilde{Y}_{1t}).$$

The case for $i = 2$ is similarly obtained as follows. For $s = 1$,

$$\Delta_{2,1}(\tilde{Y}_{1t}) = a_{21}(|\tilde{Y}_{1t}| - |Y_{1t}|) + a_{22}(|\tilde{Y}_{2t}| - |Y_{2t}|),$$

while for $s > 1$,

$$\Delta_{2,s}(\tilde{Y}_{1t}) = b_{21}\Delta_{1,s-1}(\tilde{Y}_{1t}) + b_{22}\Delta_{2,s-1}(\tilde{Y}_{1t}).$$

Now, let us define

$$\Delta_s(\tilde{Y}_{1t}) := \begin{bmatrix} \Delta_{1,s}(\tilde{Y}_{1t}) \\ \Delta_{2,s}(\tilde{Y}_{1t}) \end{bmatrix},$$

and

$$D_t := |\tilde{Y}_t| - |Y_t|. \tag{12}$$

Then, we can show that the pseudo QIRF is compactly expressed as follows:

$$\Delta_s(\tilde{Y}_{1t}) = AD_t \quad \text{for } s = 1 \tag{13}$$
$$\Delta_s(\tilde{Y}_{1t}) = B^{(s-1)}AD_t \quad \text{for } s > 1.$$

The pseudo QIRF when there is a shock (or intervention) to Y_{2t} only at time t can be analogously obtained.

It is important to be aware of two caveats in our analysis. First, if returns follow the structure in (8), shocks to ε_t will generally result in changes of Y_t which are correlated, contemporaneously and over time. In our empirical application, we take into account the contemporaneous correlation by identifying the structural shocks ε_{1t} and ε_{2t} in (8) using a standard Cholesky decomposition. However, since the DGP in (8) is not fully specified as explained before, it is not possible to take into account the impact that these structural shocks have on future returns Y_{t+s}, $s > 1$, unless one is willing to impose additional structure on the distribution of the error terms. We leave this important issue for future research.

Second, it is not straightforward to define impulse response functions for non-linear models; this issue has been discussed by Gallant et al. (1993), Potter (2000) and Lütkepohl (2008). The problem is that the impulse response for non-linear, non affine functions generally depends on the type of non-linearity, the history of past observations and on the impulse itself. This issue affects also our derivation, as shown in Eqs. (12) and (13) in which the pseudo QIRF depends on the initial value (Y_t), and is affected by the sign and magnitude of the intervention δ through the absolute function. In our implementation, we set the variable Y_t, which is originally shocked, equal to 0. Under this particular choice, the intervention δ always results in a larger value of $|\tilde{Y}_t|$ relative to the original observation $|Y_t|$, which in turn makes D_t in (12) always positive. Since the pseudo QIRFs considered in this paper are linear in D_t, the resulting impulse responses retain the standard interpretation with respect to D_t. In more general cases, however, additional care in the definition of shocks and the interpretation of the quantile impulse response functions needs to be exercised.

5.1. Standard errors for the pseudo quantile impulse response functions

Standard errors for the quantile impulse response function can be computed by exploiting the asymptotic properties of continuous transformations of random vectors (see for instance Proposition 7.4 of Hamilton, 1994). Specifically, recognizing that the above pseudo QIRF is a function of the vector of parameters $\hat{\alpha}_T$, we obtain:

$$T^{1/2}[\Delta_s(\tilde{Y}_{1t}; \hat{\alpha}_T) - \Delta_s(\tilde{Y}_{1t}; \alpha^*)] \xrightarrow{d} N(0, G_s(Q^{*-1}V^*Q^{*-1})G_s'),$$

where $G_s := \partial \Delta_s(\tilde{Y}_{1t}; \alpha)/\partial\alpha'$.

The matrix G_s can be computed analytically for $s > 1$ as follows:

$$\begin{aligned} G_s &= \partial(B^{(s-1)}AD_t)/\partial\alpha' \\ &= B^{(s-1)}\frac{\partial\, vec(AD_t)}{\partial\alpha'} + ((AD_t)' \otimes I_2)\frac{\partial\, vec(B^{(s-1)})}{\partial\alpha'}, \end{aligned}$$

where $\frac{\partial\, vec(AD_t)}{\partial\alpha'} = (D_t' \otimes I_2)\frac{\partial\, vec(A)}{\partial\alpha'}$ and $\frac{\partial\, vec(B^{(s-1)})}{\partial\alpha'} = [\sum_{i=0}^{s-2}(B')^{s-2-i} \otimes B^i]\frac{\partial\, vec(B)}{\partial\alpha'}$.

6. Empirics: assessing tail reactions of financial institutions to system wide shocks

The financial crisis which started in 2007 has had a deep impact on the conceptual thinking of systemic risk among both academics and policy makers. There has been a recognition of the shortcomings of the measures that are tailored to dealing with institution-level risks. In particular, institution-level Value at Risk measures miss important externalities associated with the need to bail out systemically important banks in order to contain potentially devastating spillovers to the rest of the economy. Therefore, government and supervisory authorities may find themselves compelled to save ex post systemically important financial institutions, while these ignore ex ante any negative externalities associated with their behavior. There exist many contributions, both theoretical and empirical, as summarized, for instance, in Brunnermeier and Oehmke (2013) or Bisias et al. (2012). For the purpose of the application we have in mind, it is useful to structure the material around two contributions, the CoVaR of Adrian and Brunnermeier (2009) and the systemic expected shortfall (SES) of Acharya et al. (2010).

Both measures aim to capture the risk of a financial institution conditional on a significant negative shock hitting another financial institution or the whole financial system. Neglecting the time t subscript for notational convenience, the $CoVaR_\theta^{j|i}$ is formally the VaR of financial institution j conditional on the return of

[6] We note that we do not consider any dynamics in the first moments of Y_t. In the subsequent empirical study, Y_t is the vector of asset returns so that imposing no dynamics in the first moment can be appropriate. To the best of our knowledge, there has been no formal and complete analysis into the issue of generalizing the proper impulse-response analysis in fully dynamic quantile models. Using a quantile autoregression framework, Koenker and Xiao (2006) allude that quantile impulse-response functions may be stochastic. In the presence of full dynamics, it can be more complicated to derive proper quantile impulse-response functions. A very rudimentary analysis is currently under way in Kim et al. (2012).

financial institution i falling below its θth-quantile (denoted by q_i^θ)[7]:

$$\Pr(Y_j < CoVaR_\theta^{j|i} | Y_i < q_i^\theta) = \theta.$$

The systemic expected shortfall is shown to be proportional to the marginal expected shortfall, which is analogously defined as:

$$MES_\theta^{j|i} = E(Y_j | Y_i < q_i^\theta).$$

The main difference with respect to CoVaR is that the expectation of Y_j conditional on Y_i being hit by a tail event, rather than just the quantile, is considered. In practice, loss distributions conditional on tail events are extremely hard to estimate. One strategy is to standardize the returns by estimated volatility or quantiles, and then apply non-parametric techniques, as done in Manganelli and Engle (2002) or Brownlees and Engle (2010). An alternative is to use the extreme value theory to impose a parametric structure on the tail behavior as done in Hartmann et al. (2004).

As we will show in the rest of this section, the theoretical framework developed in this paper lends itself to a coherent modeling of the dynamics of the tail interdependence implicit in both the CoVaR and systemic expected shortfall measures. One notable advantage of our multivariate regression quantiles framework – besides providing a robust, semi-parametric technique which does not rely on strong distributional assumptions – is that it is tailored to directly model the object of interest.

In this section, we apply our model to study the spillovers that occur in the equity return quantiles of a sample of 230 financial institution around the world by estimating a bivariate 1%-VaR model. This is a special case of the fully general MVMQ-CAViaR model in that we fix the quantile index at $\theta = 1\%$ and focus only on the multivariate aspect of the model.[8]

Theoretically, we can jointly analyze all of 230 financial institutions in our sample, but the excessive computational burden prevents the implementation of such a joint estimation. Instead, we examine bivariate models, whereby for each of these institutions, we estimate a bivariate CAViaR model where the first variable Y_{1t} is the return on a portfolio of financial institutions, and the second variable Y_{2t} is the return on the chosen financial institution. Hence, in the end, we will estimate 230 bivariate models in total. Since Y_{1t} is the return on a portfolio and Y_{2t} is the return on a specific asset, we assume that shocks to Y_{1t} are allowed to have a direct impact on Y_{2t}, but shocks to Y_{2t} do not have a direct impact on Y_{1t}. In principle, since the financial institution is part of the index, one must exclude this financial institution from the index to ensure perfect orthogonality. In practice, since our index is equally weighted and contains a large number of stocks (96 for Europe, 70 for North America and 64 for Asia; see Table 2), the inclusion of the financial institution has a negligible impact. Assuming that the θth-quantile processes for Y_{1t} and Y_{2t} follow the MVMQ-CAViaR(1, 1) model, we employ the proposed method to estimate the bivariate model.[9] Any empirical evidence for non-zero off-diagonal terms in either A or B will indicate the presence of tail-dependence between the two variables.

6.1. Data and optimization strategy

The data used in this section have been downloaded from Datastream. We considered three main global sub-indices: banks, financial services, and insurances. The sample includes daily closing prices from 1 January 2000 to 6 August 2010. Prices were transformed into continuously compounded log returns, giving an estimation sample size of 2765 observations. We use 453 additional observations up to 2 May 2012, for the out-of-sample exercises. We eliminated all the stocks whose times series started later than 1 January 2000, or which stopped after this date. At the end of this process, we were left with 230 stocks.

Table 1 reports the names of the financial institutions in our sample, together with the country of origin and the sector they are associated with, as from Datastream classification. It also reports for each financial institution the average (over the period January 2000–August 2010) market value and leverage. Leverage is provided by Datastream and is defined as the ratio of short and long debt over common equity. Table 2 shows the breakdown of the stocks by sector and by geographic area. There are twice as many financial institutions classified as banks in our sample relative to those classified as financial services or insurances. The distribution across geographic areas is more balanced, with a greater number of EU financial institutions and a slightly lower Asian representation. The proxy for the market index used in each bivariate quantile estimation is the equally weighted average of all the financial institutions in the same geographic area, in order to avoid asynchronicity issues.

We estimated 230 bivariate 1% quantile models between the market index and each of the 230 financial institutions in our sample. It is worth mentioning that an important data assumption required to estimate the bivariate CAViaR model is the stationarity condition in Assumption 1. Financial return data such as ours are well-known to be stationary whereas their levels are integrated so that the data assumption is satisfied in our application. Each model is estimated using, as starting values in the optimization routine, the univariate CAViaR estimates and initializing the remaining parameters at zero. Next, we minimized the regression quantile objective function (7) using the fminsearch optimization function in Matlab, which is based on the Nelder–Mead simplex algorithm. In calculating the standard errors, we have set the bandwidth as suggested by Koenker (2005, pp. 81) and Machado and Silva (2013). In particular, we define the bandwidth $\hat{c}_T$ as:

$$\hat{c}_T = \hat{\kappa}_T \left[\Phi^{-1}(\theta + h_T) - \Phi^{-1}(\theta - h_T) \right]$$

where h_T is defined as

$$h_T = T^{-1/3} \left(\Phi^{-1}(1 - 0.05/2) \right)^{2/3} \left(\frac{1.5 \left(\phi\left(\Phi^{-1}(\theta) \right) \right)^2}{2 \left(\Phi^{-1}(\theta) \right)^2 + 1} \right)^{1/3}$$

where $\Phi(z)$ and $\phi(z)$ are, respectively, the cumulative distribution and probability density functions of $N(0, 1)$. Following Machado and Silva (2013), we define $\hat{\kappa}_T$ as the median absolute deviation of the θth-quantile regression residuals.[10]

[7] It is straightforward to derive an estimate of the CoVaR from the model in (10). For instance, if the conditioning event C^i is defined as $Y_{2,t-1} = q_{2,t-1}$, (that is, financial institution 2 is hit by a shock equal to its quantile) the associated CoVaR for financial institution 1 is given by $q_{1,t} = c_1(\theta) + a_{11}(\theta)|Y_{1t-1}| + a_{12}(\theta)|q_{2,t-1}| + b_{11}(\theta)q_{1,t-1} + b_{12}(\theta)q_{2,t-1}$. Incidentally, this identification scheme illustrates the potential pitfalls of choosing appropriate conditioning events for the CoVaR measures. Defining the conditioning event C^i as $Y_{2,t-1} = q_{2,t-1}$, as done before, neglects the fact that shock to the financial institution 2 may be correlated with that of other financial institutions, therefore producing a potentially misleading classification of the systemic importance of financial institutions.

[8] Although it may be computationally demanding, it is possible to focus not only on the multivariate aspect, but also the multi-quantile aspect of the full model. One possibility of allowing for such a multi-quantile aspect is to consider a robust skewness measure, such as the conditional Bowley coefficient in White et al. (2008). Another possibility is to use this framework to compute the Delta CoVaR of Adrian and Brunnermeier (2009), which is the difference between the 1% quantile and the median.

[9] We note that imposing the location-scale shift specification in (9) can result in the bivariate CAViaR model in (10), but the converse is not true. Hence, assuming the bivariate CAViaR model in (10) does not necessarily imply the location-scale shift specification.

[10] The figures and tables in the paper can be replicated using the data and Matlab codes available at www.simonemanganelli.org.

Table 1
Financial institutions included in the sample.

	Name	MNEM	CTRY	SEC	MV	LEV
1	77 BANK	SSBK	JP	BK	2 294	22
2	ALLIED IRISH BANKS	ALBK	IE	BK	12 724	765
3	ALPHA BANK	PIST	GR	BK	6 916	1020
4	AUS.AND NZ.BANKING GP.	ANZX	AU	BK	27 771	444
5	AWA BANK	AWAT	JP	BK	1 314	50
6	BANK OF IRELAND	BKIR	IE	BK	11 170	905
7	BANKINTER ‘R’	BKT	ES	BK	3 985	1447
8	BARCLAYS	BARC	GB	BK	57 032	1146
9	BB&T	BBT	US	BK	18 048	209
10	BANCA CARIGE	CRG	IT	BK	3 629	427
11	BANCA MONTE DEI PASCHI	BMPS	IT	BK	9 850	859
12	BANCA POPOLARE DI MILANO	PMI	IT	BK	3 095	523
13	BANCA PPO.DI SONDRIO	BPSO	IT	BK	2 608	350
14	BANCA PPO.EMILIA ROMAGNA	BPE	IT	BK	3 397	693
15	BBV.ARGENTARIA	BBVA	ES	BK	53 390	795
16	BANCO COMR.PORTUGUES ‘R’	BCP	PT	BK	8 638	1030
17	BANCO DE VALENCIA	BVA	ES	BK	2 904	740
18	BANCO ESPIRITO SANTO	BES	PT	BK	5 455	826
19	BANCO POPOLARE	BP	IT	BK	6 441	644
20	BANCO POPULAR ESPANOL	POP	ES	BK	12 750	662
21	BANCO SANTANDER	SCH	ES	BK	73 236	702
22	BNP PARIBAS	BNP	FR	BK	63 471	700
23	BANK OF AMERICA	BAC	US	BK	142 503	363
24	BANK OF EAST ASIA	BEAA	HK	BK	5 094	88
25	BANK OF KYOTO	KYTB	JP	BK	2 692	34
26	BANK OF MONTREAL	BMO	CA	BK	20 647	256
27	BK.OF NOVA SCOTIA	BNS	CA	BK	30 637	285
28	BANK OF QLND.	BOQX	AU	BK	955	144
29	BANK OF YOKOHAMA	YOKO	JP	BK	7 081	62
30	BENDIGO & ADELAIDE BANK	BENX	AU	BK	1 284	45
31	COMMERZBANK (XET)	CBKX	DE	BK	14 330	1908
32	CREDIT SUISSE GROUP N	CSGN	CH	BK	52 691	1224
33	CREDITO VALTELLINES	CVAL	IT	BK	1 044	642
34	CANADIAN IMP.BK.COM.	CM	CA	BK	18 855	266
35	CHIBA BANK	CHBK	JP	BK	4 979	59
36	CHUGOKU BANK	CHUT	JP	BK	2 549	33
37	CHUO MITSUI TST.HDG.	SMTH	JP	BK	5 436	1836
38	CITIGROUP	C	US	BK	195 444	479
39	COMERICA	CMA	US	BK	8 053	158
40	COMMONWEALTH BK.OF AUS.	CBAX	AU	BK	36 847	378
41	DANSKE BANK	DAB	DK	BK	16 690	1527
42	DBS GROUP HOLDINGS	DBSS	SG	BK	15 398	127
43	DEUTSCHE BANK (XET)	DBKX	DE	BK	46 986	959
44	DEXIA	DEX	BE	BK	19 402	3037
45	DNB NOR	DNB	NO	BK	10 378	694
46	DAISHI BANK	DANK	JP	BK	1 440	61
47	EFG EUROBANK ERGASIAS	EFG	GR	BK	7 806	518
48	ERSTE GROUP BANK	ERS	AT	BK	10 674	1193
49	FIFTH THIRD BANCORP	FITB	US	BK	22 587	196
50	FUKUOKA FINANCIAL GP.	FUKU	JP	BK	3 713	119
51	SOCIETE GENERALE	SGE	FR	BK	42 042	641
52	GUNMA BANK	GMAB	JP	BK	2 845	40
53	HSBC HOLDINGS	HSBC	HK	BK	156 260	287
54	HACHIJUNI BANK	HABT	JP	BK	3 208	18
55	HANG SENG BANK	HSBA	HK	BK	24 971	27
56	HIGO BANK	HIGO	JP	BK	1 361	7
57	HIROSHIMA BANK	HRBK	JP	BK	2 713	119
58	HOKUHOKU FINL. GP.	HFIN	JP	BK	2 870	111
59	HUDSON CITY BANC.	HCBK	US	BK	5 827	374
60	HUNTINGTON BCSH.	HBAN	US	BK	4 569	203
61	HYAKUGO BANK	OBAN	JP	BK	1 350	21
62	HYAKUJUSHI BANK	OFBK	JP	BK	1 794	53
63	INTESA SANPAOLO	ISP	IT	BK	35 996	715
64	IYO BANK	IYOT	JP	BK	2 604	27
65	JP MORGAN CHASE & CO.	JPM	US	BK	113 168	391
66	JYSKE BANK	JYS	DK	BK	2 165	654
67	JOYO BANK	JOYO	JP	BK	3 732	48
68	JUROKU BANK	JURT	JP	BK	1 707	43
69	KBC GROUP	KB	BE	BK	22 340	587
70	KAGOSHIMA BANK	KABK	JP	BK	1 239	29
71	KEIYO BANK	CSOG	JP	BK	1 220	7
72	KEYCORP	KEY	US	BK	10 460	271
73	LLOYDS BANKING GROUP	LLOY	GB	BK	48 830	798
74	M&T BK.	MTB	US	BK	9 208	184
75	MEDIOBANCA	MB	IT	BK	10 754	577

(continued on next page)

Table 1 (*continued*)

	Name	MNEM	CTRY	SEC	MV	LEV
76	MARSHALL & ILSLEY	MI	US	BK	7 132	226
77	MIZUHO TST.& BKG.	YATR	JP	BK	6 843	1562
78	NATIONAL BK.OF GREECE	ETE	GR	BK	12 524	289
79	NATIXIS	KN@F	FR	BK	9 709	1283
80	NORDEA BANK	NDA	SE	BK	26 268	612
81	NANTO BANK	NANT	JP	BK	1 330	53
82	NATIONAL AUS.BANK	NABX	AU	BK	35 923	489
83	NAT.BK.OF CANADA	NA	CA	BK	6 175	487
84	NY.CMTY.BANC.	NYCB	US	BK	4 287	301
85	NISHI-NIPPON CITY BANK	NSHI	JP	BK	2 172	128
86	NORTHERN TRUST	NTRS	US	AM	12 419	248
87	OGAKI KYORITSU BANK	OKBT	JP	BK	1 493	78
88	OVERSEA-CHINESE BKG.	OCBC	SG	BK	12 231	148
89	BANK OF PIRAEUS	PEIR	GR	BK	4 172	646
90	PNC FINL.SVS.GP.	PNC	US	BK	18 560	179
91	POHJOLA PANKKI A	POH	FI	BK	1 502	1075
92	PEOPLES UNITED FINANCIAL	PBCT	US	BK	3 631	99
93	ROYAL BANK OF SCTL.GP.	RBS	GB	BK	72 590	619
94	REGIONS FINL.NEW	RF	US	BK	10 203	159
95	RESONA HOLDINGS	DBHI	JP	BK	15 946	1123
96	ROYAL BANK CANADA	RY	CA	BK	41 843	214
97	SEB 'A'	SEA	SE	BK	11 159	1073
98	STANDARD CHARTERED	STAN	GB	BK	27 161	323
99	SVENSKA HANDBKN.'A'	SVK	SE	BK	13 288	1397
100	SWEDBANK 'A'	SWED	SE	BK	9 828	1230
101	SYDBANK	SYD	DK	BK	1 404	620
102	SAN-IN GODO BANK	SIGB	JP	BK	1 285	46
103	SHIGA BANK	SHIG	JP	BK	1 399	30
104	SHINKIN CENTRAL BANK PF.	SKCB	JP	BK	1 372	821
105	SUMITOMO MITSUI FINL.GP.	SMFI	JP	BK	45 061	835
106	SUMITOMO TRUST & BANK.	SUMT	JP	BK	10 546	434
107	SUNTRUST BANKS	STI	US	BK	19 413	201
108	SUNCORP-METWAY	SUNX	AU	SF	6 732	259
109	SURUGA BANK	SURB	JP	BK	2 522	6
110	TORONTO-DOMINION BANK	TD	CA	BK	31 271	132
111	US BANCORP	USB	US	BK	46 133	265
112	UBS 'R'	UBSN	CH	BK	76 148	1587
113	UNICREDIT	UCG	IT	BK	47 237	695
114	UNITED OVERSEAS BANK	UOBS	SG	BK	13 924	215
115	VALIANT 'R'	VATN	CH	BK	1 643	322
116	WELLS FARGO & CO	WFC	US	BK	98 812	260
117	WESTPAC BANKING	WBCX	AU	BK	29 154	470
118	WING HANG BANK	WHBK	HK	BK	2 023	40
119	YAMAGUCHI FINL.GP.	YMCB	JP	BK	2 246	25
120	3I GROUP	III	GB	SF	7 289	61
121	ABERDEEN ASSET MAN.	ADN	GB	AM	1 274	62
122	ACKERMANS & VAN HAAREN	ACK	BE	SF	1 673	36
123	AMP	AMPX	AU	LI	10 594	316
124	ASX	ASXX	AU	IS	2 761	4
125	ACOM	ACOM	JP	CF	7 911	213
126	AMERICAN EXPRESS	AXP	US	CF	56 536	407
127	BANK OF NEW YORK MELLON	BK	US	AM	31 034	93
128	BLACKROCK	BLK	US	AM	9 237	18
129	CI FINANCIAL	CIX	CA	AM	3 263	49
130	CLOSE BROTHERS GROUP	CBG	GB	IS	1 912	193
131	CIE.NALE.A PTF.	NAT	BE	SF	4 428	48
132	CRITERIA CAIXACORP	CABK	ES	BK	17 005	127
133	CHALLENGER FINL.SVS.GP.	CGFX	AU	LI	1 026	807
134	CHARLES SCHWAB	SCHW	US	IS	21 839	27
135	CHINA EVERBRIGHT	IHDH	HK	SF	1 746	6
136	COMPUTERSHARE	CPUX	AU	FA	2 880	76
137	CREDIT SAISON	SECR	JP	CF	4 428	405
138	DAIWA SECURITIES GROUP	DS@N	JP	IS	11 452	526
139	EURAZEO	ERF	FR	SF	3 798	119
140	EATON VANCE NV.	EV	US	AM	3 009	88
141	EQUIFAX	EFX	US	SF	4 028	152
142	FRANKLIN RESOURCES	BEN	US	AM	17 121	17
143	GAM HOLDING	GAM	CH	AM	6 116	101
144	GBL NEW	GBLN	BE	SF	11 164	8
145	GOLDMAN SACHS GP.	GS	US	IS	56 514	752
146	ICAP	IAP	GB	IS	3 359	27
147	IGM FINL.	IGM	CA	AM	7 608	46
148	INDUSTRIVARDEN 'A'	IU	SE	SF	3 053	26
149	INTERMEDIATE CAPITAL GP.	ICP	GB	SF	1 313	201
150	KINNEVIK 'B'	KIVB	SE	SF	1 754	75
151	INVESTOR 'B'	ISBF	SE	SF	6 807	28

(*continued on next page*)

Table 1 (*continued*)

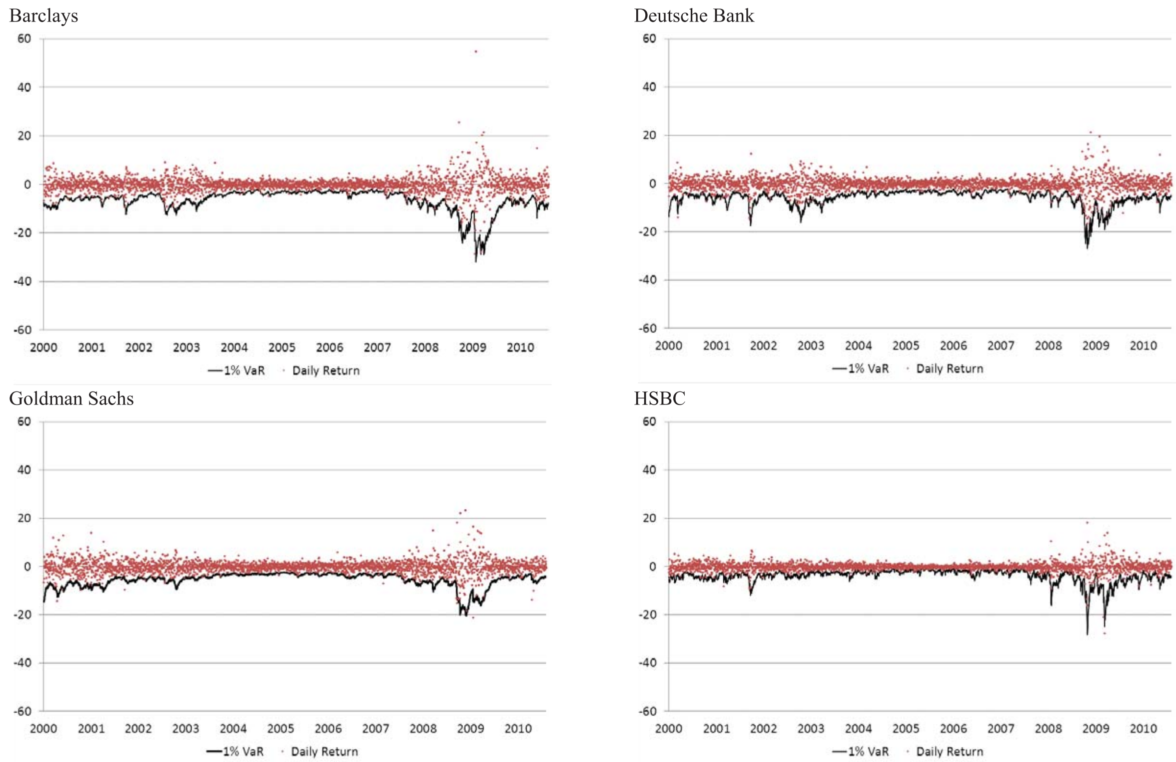
	Name	MNEM	CTRY	SEC	MV	LEV
152	LEGG MASON	LM	US	AM	6 342	41
153	MAN GROUP	EMG	GB	AM	8 969	44
154	MARFIN INV.GP.HDG.	INT	GR	SF	1 843	69
155	MACQUARIE GROUP	MQG	AU	IS	8 431	767
156	MITSUB.UFJ LSE.& FINANCE	DIML	JP	SF	1 983	1379
157	MIZUHO SECURITIES	NJPS	JP	IS	2 849	673
158	MOODY'S	MCO	US	SF	9 359	168
159	MORGAN STANLEY	MS	US	IS	58 285	1018
160	NOMURA HDG.	NM@N	JP	IS	30 838	782
161	ORIX	ORIX	JP	SF	11 756	703
162	PARGESA 'B'	PARG	CH	SF	5 168	28
163	PROVIDENT FINANCIAL	PFG	GB	CF	2 623	250
164	PERPETUAL	PPTX	AU	AM	1 383	45
165	RATOS 'B'	RTBF	SE	SF	1 693	55
166	SCHRODERS	SDR	GB	AM	3 641	16
167	SLM	SLM	US	CF	13 762	3477
168	SOFINA	SOF	BE	SF	2 537	2
169	STATE STREET	STT	US	AM	18 719	390
170	T ROWE PRICE GP.	TROW	US	AM	8 380	15
171	TD AMERITRADE HOLDING	AMTD	US	IS	6 486	40
172	WENDEL	MF@F	FR	SF	3 513	315
173	ACE	ACE	US	PCI	13 111	28
174	AEGON	AGN	NL	LI	26 400	81
175	AFLAC	AFL	US	LI	19 805	28
176	AGEAS (EX-FORTIS)	AGS	BE	LI	29 250	1005
177	ALLIANZ (XET)	ALV	DE	FLI	61 436	585
178	AMLIN	AML	GB	PCI	1 602	19
179	AON	AON	US	IB	9 825	50
180	GENERALI	G	IT	FLI	40 022	111
181	AVIVA	AV.	GB	LI	25 396	95
182	AXA ASIA PACIFIC HDG.	AXAX	AU	LI	5 778	45
183	AXA	MIDI	FR	FLI	51 760	60
184	ALLSTATE	ALL	US	PCI	27 680	29
185	AMERICAN INTL.GP.	AIG	US	FLI	138 736	410
186	ARCH CAP.GP.	ACGL	US	PCI	2 382	12
187	BALOISE-HOLDING AG	BALN	CH	FLI	4 015	101
188	BERKSHIRE HATHAWAY 'B'	BRKB	US	RE	30 983	22
189	CNP ASSURANCES	CNP	FR	LI	10 038	102
190	CHUBB	CB	US	PCI	15 311	25
191	CINCINNATI FINL.	CINF	US	PCI	6 277	26
192	EVEREST RE GP.	RE	US	RE	4 391	26
193	FAIRFAX FINL.HDG.	FFH	CA	PCI	3 013	68
194	GREAT WEST LIFECO	GWO	CA	LI	17 341	45
195	HANNOVER RUCK. (XET)	HNR1	DE	RE	3 838	50
196	HELVETIA HOLDING N	HEPN	CH	FLI	1 802	11
197	HARTFORD FINL.SVS.GP.	HIG	US	FLI	17 070	36
198	ING GROEP	ING	NL	LI	58 049	794
199	JARDINE LLOYD THOMPSON	JLT	GB	IB	1 564	31
200	LEGAL & GENERAL	LGEN	GB	LI	12 162	57
201	LINCOLN NAT.	LNC	US	LI	9 677	38
202	LOEWS	L	US	PCI	14 101	58
203	MAPFRE	MAP	ES	FLI	5 059	42
204	MS&AD INSURANCE GP.HDG.	MSAD	JP	PCI	11 765	15
205	MUENCHENER RUCK. (XET)	MUV2	DE	RE	34 913	36
206	MANULIFE FINANCIAL	MFC	CA	LI	30 007	44
207	MARKEL	MKL	US	PCI	2 872	52
208	MARSH & MCLENNAN	MMC	US	IB	20 242	66
209	OLD MUTUAL	OML	GB	LI	9 499	55
210	PRUDENTIAL	PRU	GB	LI	23 335	197
211	PARTNERRE	PRE	US	RE	3 417	19
212	POWER CORP.CANADA	POW	CA	LI	8 636	91
213	POWER FINL.	PWF	CA	LI	15 644	76
214	PROGRESSIVE OHIO	PGR	US	PCI	13 604	36
215	QBE INSURANCE GROUP	QBEX	AU	RE	10 440	49
216	RSA INSURANCE GROUP	RSA	GB	FLI	6 902	45
217	RENAISSANCERE HDG.	RNR	US	RE	2 934	19
218	SAMPO 'A'	SAMA	FI	PCI	8 669	36
219	SCOR SE	SCO	FR	RE	2 395	54
220	STOREBRAND	STB	NO	FLI	2 292	193
221	SWISS LIFE HOLDING	SLHN	CH	LI	5 522	99
222	SWISS RE 'R'	RUKN	CH	RE	24 129	
223	TOPDANMARK	TOP	DK	PCI	1 703	96
224	TORCHMARK	TMK	US	LI	4 901	30
225	TRAVELERS COS.	TRV	US	PCI	20 617	35
226	UNUM GROUP	UNM	US	LI	6 169	40
227	VIENNA INSURANCE GROUP A	WNST	AT	FLI	3 254	42

(*continued on next page*)

Table 1 (*continued*)

	Name	MNEM	CTRY	SEC	MV	LEV
228	W R BERKLEY	WRB	US	PCI	3 609	45
229	XL GROUP	XL	US	PCI	9 148	31
230	ZURICH FINANCIAL SVS.	ZURN	CH	FLI	28 299	53

Note: The abbreviation for the sector classification are as follows: BK = Bank, AM = Asset Management, SF = Specialty Finance, IS = Investment Service, CF = Consumer Finance, FA = Financial Administration, LI = Life Insurance, PCI = Property and Casualty Insurance, FLI = Full Line Insurance, IB = Insurance Broker, RE = Reinsurance. The average market value (MV) in million $ and leverage (LEV) over the period 2000–2010 are also reported. The leverage is computed as short and long term debt over common equity. Classification as provided by Datastream.



Note: The four charts report the in-sample 1% daily Value at Risk (VaR) for four selected institutions from 1 January 2000 to 6 August 2010, together with daily returns. The VaR is computed from a bivariate VAR for VaR, where the first equation contains the quantile of the regional index and the second equation the quantile of the individual financial institution. The empirical specification follows equation (8) in the paper: $q_t = c + A|Y_{t-1}| + Bq_{t-1}$. The crisis following the Lehman default is clearly visible in the plots.

Fig. 1. 1% quantile for selected financial institutions.

6.2. Results

Table 3 reports, as an example, the estimation results for four well-known financial institutions from different geographic areas: Barclays, Deutsche Bank, Goldman Sachs and HSBC. The diagonal autoregressive coefficients for the *B* matrix are around 0.90 and all of them are statistically significant,[11] which indicates the VaR processes are significantly autocorrelated. These findings are consistent with what is typically found in the literature using CAViaR models. Notice, however, that some of the non-diagonal coefficients for the *A* or *B* matrices are significantly different from zero. This is the case for Barclays, Goldman Sachs, and HSBC and the examples illustrate how the multivariate quantile model can uncover dynamics that cannot be detected by estimating univariate quantile models. In general, we reject the joint null hypothesis that all off-diagonal coefficients of the matrices *A* and *B* are equal to zero at the 5% level for around 100 financial institutions out of the 230 in our sample. The resulting estimated 1% quantiles for Barclays, Deutsche Bank, Goldman Sachs and HSBC are reported in Fig. 1. The quantile plots clearly reveal the generalized sharp increase in risk following the Lehman bankruptcy. Careful inspection of the plots also reveals a noticeable cross-sectional difference, with the risk for Goldman Sachs being contained to about two thirds of the risk of Barclays at the height of the crisis.

Table 4 reports summary statistics for the full cross-section of coefficients. Average values are in line with the values reported in Table 3. For instance, the autoregressive coefficient for b_{11} and b_{22} are 0.84 and 0.86 respectively. At the same time, the cross-sectional standard deviation and the min–max range reveal quite substantial heterogeneity in the estimates.

Table 5 provides an assessment of the overall performance of the 230 estimated bivariate models. The performance is based on the number of VaR exceedances both in-sample and out-of-sample. Specifically, for each of the 230 bivariate VAR for VaR models, the time series of returns is transformed into a time series of indicator functions which take value one if the return

[11] It is noted that the standard errors in Tables 3 and 4 have been computed using the asymptotic distribution result in Theorem 2. As explained in Section 3, if readers are concerned about the extreme value theory, then those standard errors should be adjusted following the procedure in Chernozhukov and Fernandez-Val (2011). The feasible inference methodology for extremal quantile model proposed in Chernozhukov and Fernandez-Val (2011) is based on a linear quantile model while our proposed model is nonlinear. Nonetheless, we conjecture that the procedure may be still applicable with some slight modifications, but some non-trivial complications might arise. A formal investigation is left for further research.

Table 2
Breakdown of financial institutions by sector and by geographic area.

		Banks	Financial services					Total	Insurance					Total	Total
			Asset management	Specialty finance	Investment service	Consumer finance	Financial administration		Life insurance	Property and casualty insurance	Full line insurance	Insurance broker	Re-insurance		
Europe	AT	1	0	0	0	0	0	**0**	0	0	1	0	0	**1**	***2***
	BE	2	0	4	0	0	0	**4**	1	0	0	0	0	**1**	***7***
	DE	2	0	0	0	0	0	**0**	0	0	1	0	2	**3**	***5***
	DK	3	0	0	0	0	0	**0**	0	1	0	0	0	**1**	***4***
	CH	3	1	1	0	0	0	**2**	1	0	3	0	1	**5**	***10***
	ES	6	0	0	0	0	0	**0**	0	0	1	0	0	**1**	***7***
	FI	1	0	0	0	0	0	**0**	0	1	0	0	0	**1**	***2***
	FR	3	0	2	0	0	0	**2**	1	0	1	0	1	**3**	***8***
	GB	4	3	2	2	1	0	**8**	4	1	1	1	0	**7**	***19***
	GR	4	0	1	0	0	0	**1**	0	0	0	0	0	**0**	***5***
	IE	2	0	0	0	0	0	**0**	0	0	0	0	0	**0**	***2***
	IT	10	0	0	0	0	0	**0**	0	0	1	0	0	**1**	***11***
	NL	0	0	0	0	0	0	**0**	2	0	0	0	0	**2**	***2***
	NO	1	0	0	0	0	0	**0**	0	0	1	0	0	**1**	***2***
	PT	2	0	0	0	0	0	**0**	0	0	0	0	0	**0**	***2***
	SE	4	0	4	0	0	0	**4**	0	0	0	0	0	**0**	***8***
Total		**48**	**4**	**14**	**2**	**1**	**0**	**21**	**9**	**3**	**10**	**1**	**4**	**27**	***96***
North America	CA	6	2	0	0	0	0	**2**	4	1	0	0	0	**5**	***13***
	US	18	8	2	4	2	0	**16**	4	11	2	2	4	**23**	***57***
Total		**24**	**10**	**2**	**4**	**2**	**0**	**18**	**8**	**12**	**2**	**2**	**4**	**28**	***70***
Asia	AU	6	1	1	2	0	1	**5**	3	0	0	0	1	**4**	***15***
	HK	4	0	1	0	0	0	**1**	0	0	0	0	0	**0**	***5***
	JP	33	0	2	3	2	0	**7**	0	1	0	0	0	**1**	***41***
	SG	3	0	0	0	0	0	**0**	0	0	0	0	0	**0**	***3***
Total		**46**	**1**	**4**	**5**	**2**	**1**	**13**	**3**	**1**	**0**	**0**	**1**	**5**	***64***
Total		***118***	***15***	***20***	***11***	***5***	***1***	***52***	***20***	***16***	***12***	***3***	***9***	***60***	***230***

Note: Classification as provided by datastream.

Table 3
Estimates and standard errors for selected financial institutions.

Barclays				
c_1	a_{11}	a_{12}	b_{11}	b_{12}
−0.15 ***	−0.48 ***	−0.05 ***	0.82 ***	−0.01 **
0.05	*0.12*	*0.01*	*0.05*	*0.01*
c_2	a_{21}	a_{22}	b_{21}	b_{22}
−0.10 **	−0.30 ***	−0.15 ***	−0.12 **	0.96 ***
0.05	*0.10*	*0.05*	*0.05*	*0.01*
Deutsche Bank				
c_1	a_{11}	a_{12}	b_{11}	b_{12}
−0.12 *	−0.36 **	−0.07	0.88 ***	−0.03
0.07	*0.15*	*0.07*	*0.06*	*0.02*
c_2	a_{21}	a_{22}	b_{21}	b_{22}
−0.16 **	−0.06	−0.34	0.00	0.86 ***
0.07	*0.26*	*0.25*	*0.10*	*0.08*
Goldman Sachs				
c_1	a_{11}	a_{12}	b_{11}	b_{12}
−0.04 *	−0.19 **	−0.08 ***	0.93 ***	−0.03 **
0.02	*0.09*	*0.02*	*0.03*	*0.01*
c_2	a_{21}	a_{22}	b_{21}	b_{22}
−0.03	0.00	−0.16 **	0.01	0.94 ***
0.02	*0.11*	*0.07*	*0.04*	*0.03*
HSBC				
c_1	a_{11}	a_{12}	b_{11}	b_{12}
−0.09	−0.29 **	−0.06	0.89 ***	−0.02
0.09	*0.12*	*0.13*	*0.07*	*0.04*
c_2	a_{21}	a_{22}	b_{21}	b_{22}
−0.14	−0.49	−0.40	−0.16 *	0.87 ***
0.15	*0.45*	*0.36*	*0.09*	*0.09*

Note: Estimated coefficients are in the first row. Standard errors are reported in italic in the second row. The coefficients correspond to the VAR for VaR model reported in Eq. (8) of the paper.
[*] Denotes coefficients significant at the 10% confidence level.
[**] Denotes coefficients significant at the 5% confidence level.
[***] Denotes coefficients significant at the 1% confidence level.

exceeds the VaR and zero otherwise. When estimating a 1% VaR, on average one should expect stock market returns to exceed the VaR 1% of the times. The first line of the table reveals that the in-sample estimates are relatively precise, as shown by the accurate average and median number of exceedances, their very low standard deviations and the relatively narrow cross-sectional min–max range. The out-of-sample performance is less accurate, as to be expected, with substantially higher standard deviations and very large min–max range. The out-of-sample performance has also been assessed by applying the out-of-sample DQ test of Engle and Manganelli (2004), which tests not only whether the number of exceedances is close to the VaR confidence level, but also whether these exceedances are not correlated over time. The test reveals that the performance of the out-of-sample VaR is not rejected at a 5% confidence level for more than half of the stocks. Note that for the out-of-sample exercise the coefficients are held fixed at their estimated in-sample values.

The methodology introduced in this paper, however, allows us to go beyond the analysis of the univariate quantiles, and directly looks at the tail codependence between financial institutions and the market index. Fig. 2 displays the impulse response of the risks (and associated 95% confidence intervals) of the four financial institutions to a 2 standard deviation shock to the market index (see the discussion in Section 5 for a detailed explanation of how the pseudo impulse-response functions are computed). The horizontal axis measures the time (expressed in days), while the vertical axis measures the change in the 1% quantiles of the individual financial institutions (expressed in percentage returns) as a reaction to the market shock. The pseudo impulse response functions track how this shock propagates through the system and how long it takes to absorb it. The shock is completely reabsorbed after the pseudo impulse response function has converged again to zero.

A closer look at the pseudo impulse response functions of the four selected financial institutions reveals a few differences in how their long run risks react to shocks. For instance, Deutsche Bank and HSBC have a similar pseudo impulse response function, although HSBC's is not statistically different from zero. Goldman Sachs quantiles, instead, exhibit very little tail codependence with the market, and not statistically significant, as illustrated by the error bands straddling the zero line.

It should be borne in mind that each of the 230 bivariate models is estimated using a different information set (as the time series of the index and of a different financial institution is used for each estimation). Therefore, each pair produces a different estimate of the VaR of the index, simply because we condition on a different information set. Moreover, the coefficients and any quantities derived from them, such as pseudo impulse responses, are information set-specific. This means that naive comparisons across bivariate pairs can be misleading and are generally unwarranted. The proper context for comparing sensitivities and pseudo impulse responses is in a multivariate setting using a common information set. Because of the non trivial computational challenges involved, we leave this for future study.

These important caveats notwithstanding, averaging across the bivariate results can still provide useful summary information and suggest general features of the results. Accordingly, Fig. 3

Table 4
Summary statistics of the full cross section of coefficients.

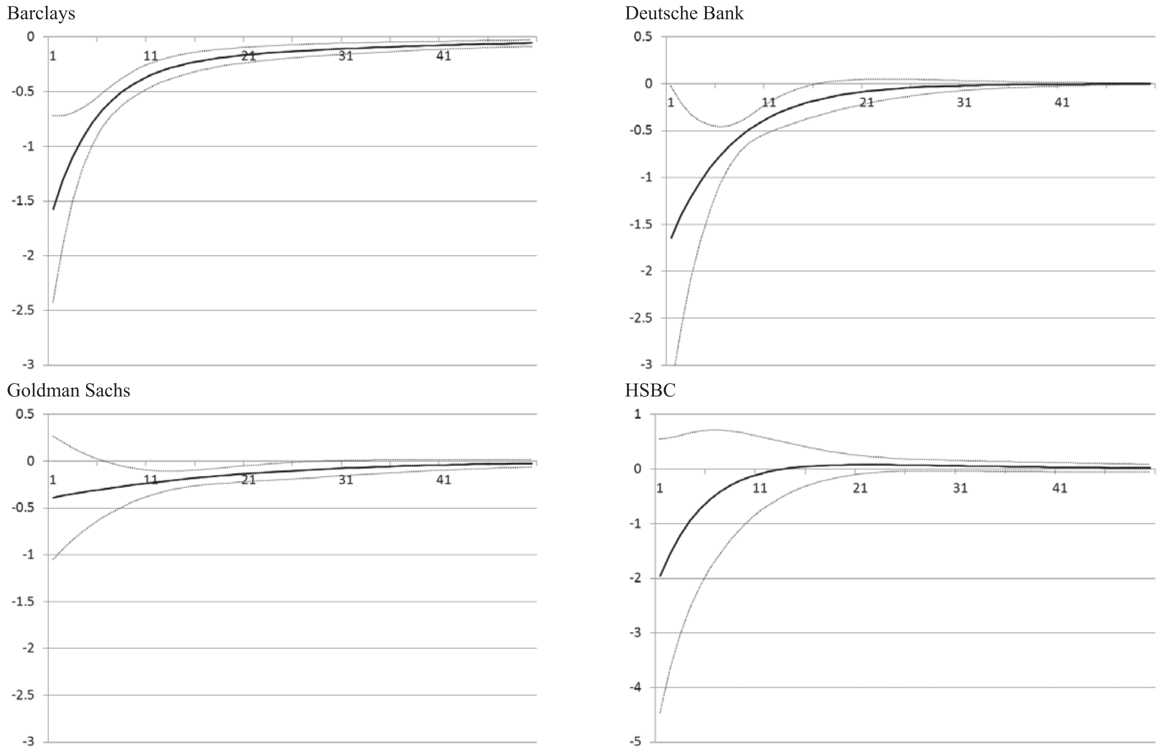
	c_1	a_{11}	a_{12}	b_{11}	b_{12}
Average	−0.07	−0.32	−0.02	0.84	0.02
std. dev.	0.17	0.13	0.07	0.20	0.12
min	−0.98	−0.70	−0.32	−0.79	−0.34
max	1.56	0.03	0.14	1.28	0.88
	c_2	a_{21}	a_{22}	b_{21}	b_{22}
Average	−0.16	−0.18	−0.24	0.02	0.86
std. dev.	0.31	0.23	0.22	0.21	0.16
min	−3.49	−1.12	−1.91	−1.11	−0.03
max	0.82	0.62	0.09	1.39	1.49

Note: The table reports the summary statistics of the coefficient estimates of the 230 bivariate VAR for VaR models. The table reveals quite substantial heterogeneity in the estimates.

Table 5
Performance evaluation.

	Average	Median	std. dev.	min	max	# stocks passing DQ test
In-sample	1.00%	1.01%	0.07%	0.25%	1.45%	–
Out-of-sample	1.33%	0.88%	5.81%	0.00%	87.64%	123

Note: The table reports the summary statistics of VaR performance evaluation, based on the number of VaR exceedances both in-sample and out-of-sample. For each of the 230 bivariate VAR for VaR models, the time series of returns is transformed into a time series of indicator functions which take value one if the return exceeds the VaR and zero otherwise. When estimating a 1% VaR, on average one should expect stock market returns to exceed the VaR 1% of the times. The first line reveals that the in-sample estimates are relatively precise, as shown by the accurate average and median, the very low standard deviations and the relatively narrow min–max range. The out-of-sample performance is less accurate, as to be expected, with substantially higher standard deviation and very large min–max range. The out-of-sample performance has been assessed also applying the out-of-sample DQ test of Engle and Manganelli (2004), which tests not only whether the number of exceedances is close to the VaR confidence level, but also that these exceedances are not correlated over time. The test reveals that the performance of the out-of-sample VaR is not rejected at a 5% confidence level for more than half of the stocks. Note that for the out-of-sample exercise the coefficients are held fixed at their estimated in-sample values.



Note: The four charts report the quantile impulse-response functions to a shock to the market for four selected institutions, together with 95% confidence intervals. The impulse-responses are derived from a bivariate VAR for VaR, where the first equation contains the quantile of the regional index and the second equation the quantile of the individual financial institution. The identification of the market shock relies on a Choleski decomposition of the daily returns, which implicitly assumes that shocks to the market can simultaneously affect the regional index and the individual financial institution, while shocks to the financial institution can affect the market only with a lag.

Fig. 2. Impulse-response functions to a shock to the market for selected financial institutions.

plots the average pseudo impulse-response functions $\Delta_{1,s}(\tilde{\varepsilon}_{2t})$ and $\Delta_{2,s}(\tilde{\varepsilon}_{1t})$ measuring the impact of a two standard deviation individual financial institution shock on the index and the impact of a two standard deviation shock to the index on the individual financial institution's risk. In the left column, the average is taken with respect to the geographical distribution. That is, the average pseudo impulse-response for the four largest euro area countries, for example, is obtained by averaging all the pseudo impulse-response functions for the German, French, Italian and Spanish financial institutions. We notice two things. First, the impact of a shock to the index (charts in the top row) is much stronger than the impact of a shock to the individual financial institution (charts in the bottom row). This result is partly driven by our identification assumption that shocks to the index have a contemporaneous impact on the return of the single financial institutions, while the institution's specific shocks have only a lagged impact on the global financial index. Second, we notice that the risk of Japanese financial institutions appears to be on average somewhat less sensitive to global shocks than their European and North American counterparts.

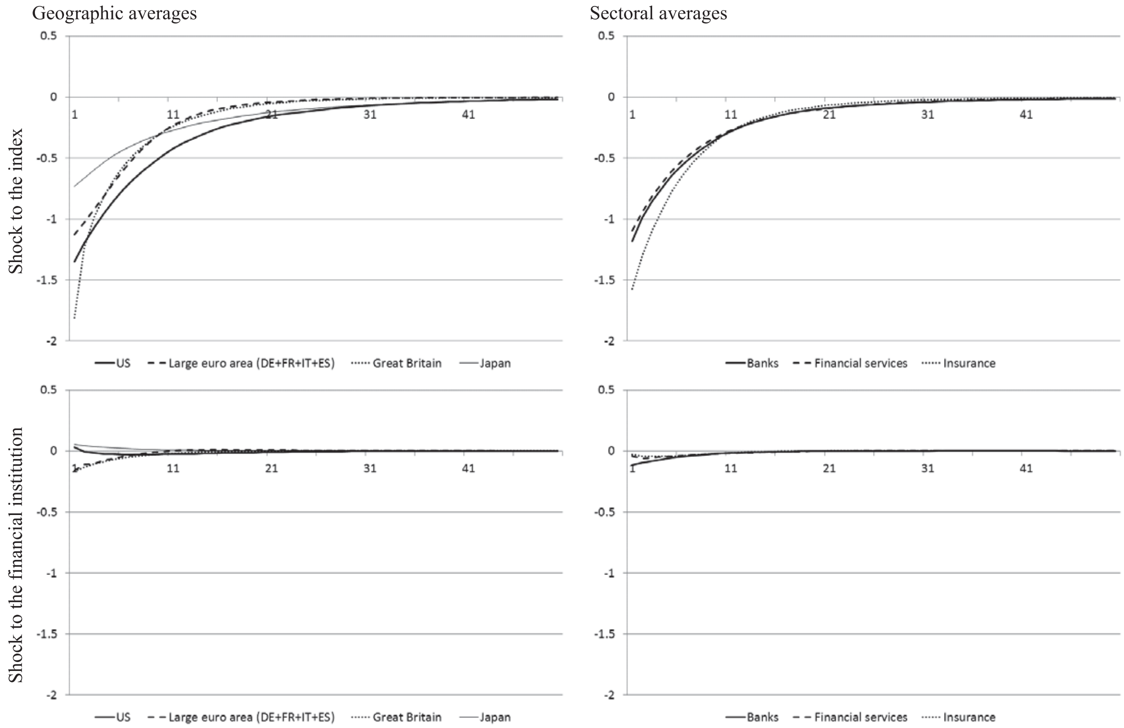
The charts on the right column of Fig. 3 plot the average pseudo impulse-response functions for the financial institutions grouped by line of business, i.e. banks, financial services, and insurances. We see that a shock to the index has a stronger initial impact on the group of insurance companies.

Two interesting dimensions along which pseudo impulse response functions can be aggregated are size and leverage, as reported in Table 1. Fig. 4 plots the average pseudo impulse-responses to a market shock for the 30 largest and smallest financial institutions, together with those of the largest and smallest leverage. It is clear that the shocks to the index have a much greater impact on the largest and most leveraged financial institutions. A two standard deviation shock to the index produces an average initial increase in the daily VaR of the largest financial institutions of about 1.7% and for the most leveraged of about 1.4%. This compares to an average increase in VaR of around 0.9% for the 30 smallest and least leveraged financial institutions. Interestingly, there is little overlap between the two groups of stocks.

To gauge to what extent the model correctly identifies the financial institutions whose risks are most exposed to market shocks, Fig. 5 plots the average quantiles of the two sets of financial institutions identified in Fig. 4. Specifically, the charts in the top panels of the figure, track the estimated in-sample quantiles developments of the 30 largest/smallest and most/least leveraged financial institutions. The charts in the bottom panels replicate the same exercise with the out-of-sample data.

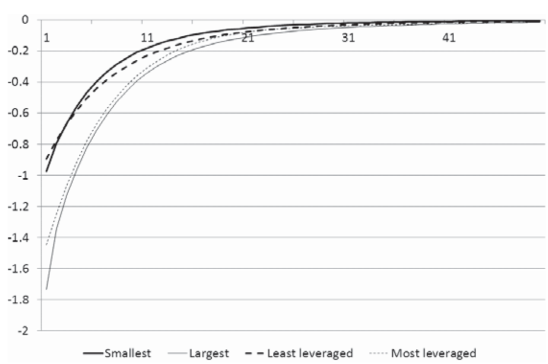
The figure presents two striking facts. First, during normal times, i.e. between 2004 and mid-2007, the quantiles of the largest/smallest financial institutions are roughly equal. Actually, there are some periods in 2003 in which the quantiles of the smallest financial institutions exceeded the quantiles of the largest ones. The second striking fact is that the situation changes abruptly in periods of market turbulence. For instance, at the beginning of the sample, in 2001–2003, the quantiles of the largest financial institutions increased significantly more than that of the smallest ones. The change in behavior during crisis periods is even more striking from 2008 onwards, showing a greater exposure to common shocks. The bottom panels reveal that similar results hold for the out-of-sample period. Of particular notice is the sharp drop in the out-of-sample quantile for the group of the largest financial institutions which occurred on 12 August, 2011, the beginning of the second phase of the euro area sovereign debt crisis.

This application illustrates how the proposed methodology can usefully inform policy makers by helping identify the set of



Note: The four charts report the average quantile impulse-response functions. Averages are taken along the geographic and sectoral dimension. The first row is the average impulse response of financial institutions' quantiles to a shock to the market. The second row is the average impulse response of markets' quantiles to a shock to the individual financial institutions. As usual, the impulse-responses are derived from a bivariate VAR for VaR, where the first equation contains the quantile of the regional index and the second equation the quantile of the individual financial institution. The first row reveals that shocks to European financial institutions are absorbed relatively quicker than in Japan or the US. The risk of insurance companies is on average more sensitive to market shocks than financial institutions in the banking and financial services sectors. Market risk reactions to shocks to individual financial institutions are more muted.

Fig. 3. Impulse-response functions by sectoral and geographic aggregation.



Note: The figure reports the average impulse-response function of the 30 financial institutions with the largest and smallest market capitalisation and leverage. Largest and most leveraged financial institution display a relatively similar risk response to market shocks, despite only four institutions belong to both groups. The average reaction is about twice the impulse-responses of smallest and least leverage banks.

Fig. 4. Impulse-response functions for institutions sorted by size and leverage.

financial institutions which may be most exposed to common shocks, especially in times of crisis. Of course, this should only be considered as a partial model-based screening device for the identification of the most systemic banks. Further analysis, market intelligence and sound judgment are other necessary elements to produce a reliable risk assessment method for the larger and more complex financial groups.

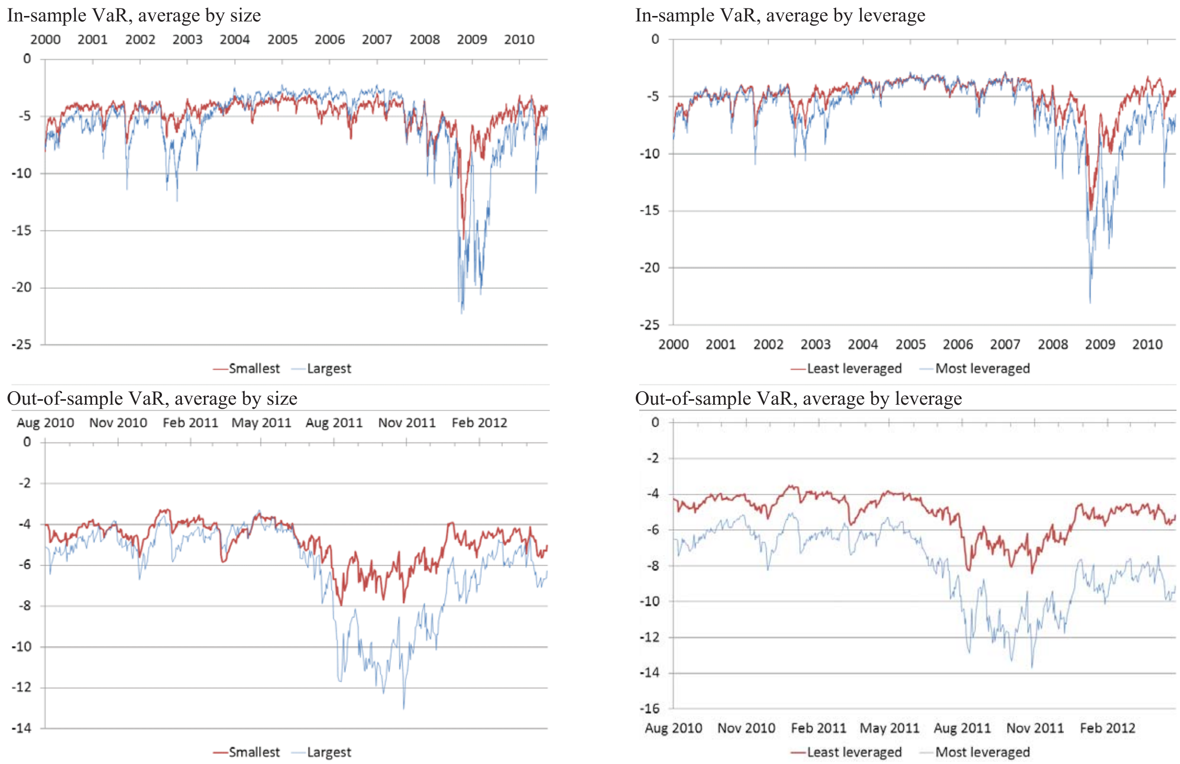
Again, we emphasize that the results presented in these figures merely summarize the pattern of the results found in the bivariate analysis of our 230 financial institutions. Cross-comparisons could be improved by estimating for instance a 3- or 4- or n-variate system using a common information set, or adopting an appropriate factor structure which would minimize the number of parameters to be estimated. Alternatively, one could impose that the B matrix in (11) is diagonal, which would be equivalent to assuming that the parameters of the system are variation free, as in Engle et al. (1983). This assumption would have the added advantage of allowing a separate estimation of each quantile. That is, for an n-variate system, the optimization problem in (7) can be broken down into n independent optimization problems, which in turn would considerably increase the computational tractability.

7. Conclusion

We have developed a theory ensuring the consistency and asymptotic normality of multivariate and multi-quantile models. Our theory is general enough to comprehensively cover models with multiple random variables, multiple confidence levels and multiple lags of the quantiles.

We conducted an empirical analysis in which we estimate a vector autoregressive model for the Value at Risk – VAR for VaR – using returns of individual financial institutions from around the world. By examining the pseudo impulse-response functions, we can study the financial institutions' long run risk reactions to shocks to the overall index. Judging from our bivariate models, we found that the risk of Asian financial institutions tends to be less sensitive to system wide shocks, whereas insurance companies exhibit a greater sensitivity to global shocks. We also found wide differences on how financial institutions react to different shocks. Both in-sample and out-of-sample analyses reveal that largest and most leveraged financial institutions are those whose risk increases the most in periods of market turbulence.

The methods developed in this paper can be applied to many other contexts. For instance, many stress-test models are built from vector autoregressive models on credit risk indicators and



Note: The figure reports the average VaR of the 30 financial institutions with the largest and smallest market capitalisation and leverage. All groups of financial institutions have similar VaR during tranquil periods. Between 2004 and 2007, largest financial institutions had actually a lower average VaR than smallest ones. In times of crisis, however, the VaR of the largest and most leveraged financial institutions increases much more than for the small and least leverage ones. This behaviour is consistent with the largest and most leveraged financial institutions having a stronger quantile impulse-response function, as illustrated in figure 4. The out-of-sample plots are computed using the estimated in-sample coefficients and reveal similar patterns to those identified in-sample.

Fig. 5. In-sample and out-of-sample average VaR by size and leverage.

macroeconomic variables. Starting from the estimated mean and adding assumptions on the multivariate distribution of the error terms, one can deduce the impact of a macro shock on the quantile of the credit risk variables. Our methodology provides a convenient alternative for stress testing, by allowing researchers to estimate vector autoregressive processes directly on the quantiles of interest, rather than on the mean.

Appendix

We establish the consistency of $\hat{\alpha}_T$ by applying the results of White (1994). For this, we impose the following moment and domination conditions. In stating this next condition and where convenient elsewhere, we exploit stationarity to omit explicit reference to all values of t.

Assumption 5. (i) For $i = 1, \ldots, n,\ E|Y_{it}| < \infty$; (ii) Let us define

$$D_{0,t} := \max_{i=1,\ldots,n} \max_{j=1,\ldots,p} \sup_{\alpha \in \mathbb{A}} |q_{i,j,t}(\cdot, \alpha)|.$$

Then $E(D_{0,t}) < \infty$.

Proof of Theorem 1. We verify the conditions of Corollary 5.11 of White (1994), which delivers $\hat{\alpha}_T \to \alpha^*$, where

$$\hat{\alpha}_T := \arg\max_{\alpha \in \mathbb{A}} T^{-1} \sum_{t=1}^{T} \varphi_t(Y_t, q_t(\cdot, \alpha)),$$

and $\varphi_t(Y_t, q_t(\cdot, \alpha)) := -\{\sum_{i=1}^{n} \sum_{j=1}^{p} \rho_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot, \alpha))\}$. Assumption 1 ensures White's Assumption 2.1. Assumption 3(i) ensures White's Assumption 5.1. Our choice of $\rho_{\theta_{ij}}$ satisfies White's Assumption 5.4. To verify White's Assumption 3.1, it suffices that $\varphi_t(Y_t, q_t(\cdot, \alpha))$ is dominated on $\mathbb{A}$ by an integrable function (ensuring White's Assumption 3.1(a), (b)), and that for each α in $\mathbb{A}$, $\{\varphi_t(Y_t, q_t(\cdot, \alpha))\}$ is stationary and ergodic (ensuring White's Assumption 3.1(c), the strong uniform law of large numbers (ULLN)). Stationarity and ergodicity is ensured by Assumptions 1 and 3(i). To show domination, we write

$$\begin{aligned} |\varphi_t(Y_t, q_t(\cdot, \alpha))| &\le \sum_{i=1}^{n} \sum_{j=1}^{p} |\rho_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot, \alpha))| \\ &= \sum_{i=1}^{n} \sum_{j=1}^{p} |(Y_{it} - q_{i,j,t}(\cdot, \alpha))(\theta_{ij} - 1_{[Y_{it} - q_{i,j,t}(\cdot,\alpha) \le 0]})| \\ &\le 2 \sum_{i=1}^{n} \sum_{j=1}^{p} (|Y_{it}| + |q_{i,j,t}(\cdot, \alpha)|) \\ &\le 2p \sum_{i=1}^{n} |Y_{it}| + 2np|D_{0,t}|, \end{aligned}$$

so that

$$\sup_{\alpha \in \mathbb{A}} |\varphi_t(Y_t, q_t(\cdot, \alpha))| \le 2p \sum_{i=1}^{n} |Y_{it}| + 2np|D_{0,t}|.$$

Thus, $2p \sum_{i=1}^{n} |Y_{it}| + 2np|D_{0,t}|$ dominates $|\varphi_t(Y_t, q_t(\cdot, \alpha))|$; this has finite expectation by Assumption 5(i), (ii).

White's Assumption 3.2 remains to be verified; here, this is the condition that α^* is the unique maximizer of $E(\varphi_t(Y_t, q_t(\cdot, \alpha)))$. Given Assumptions 2(ii.b) and 4(i), it follows through the argument that directly parallels to that of the proof by White (1994, Corollary 5.11) that for all $\alpha \in \mathbb{A}$,

$$E(\varphi_t(Y_t, q_t(\cdot, \alpha))) \le E(\varphi_t(Y_t, q_t(\cdot, \alpha^*))).$$

Thus, it suffices to show that the above inequality is strict for $\alpha \neq \alpha^*$. Consider $\alpha \neq \alpha^*$ such that $\|\alpha - \alpha^*\| > \epsilon$, and let $\Delta(\alpha) := \sum_{i=1}^n \sum_{j=1}^p E(\Delta_{i,j,t}(\alpha))$ with $\Delta_{i,j,t}(\alpha) := \rho_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\alpha)) - \rho_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\alpha^*))$. It will suffice to show that $\Delta(\alpha) > 0$. First, we define the "error" $\varepsilon_{i,j,t} := Y_{it} - q_{i,j,t}(\cdot,\alpha^*)$ and let $f_{i,j,t}(\cdot)$ be the density of $\varepsilon_{i,j,t}$ conditional on $\mathcal{F}_{t-1}$. Noting that $\delta_{i,j,t}(\alpha,\alpha^*) := q_{i,j,t}(\cdot,\alpha) - q_{i,j,t}(\cdot,\alpha^*)$, we next can show through some algebra and Assumption 2(ii.a) that

$$
\begin{aligned}
E(\Delta_{i,j,t}(\alpha)) &= E\left[\int_0^{\delta_{i,j,t}(\alpha,\alpha^*)} (\delta_{i,j,t}(\alpha,\alpha^*) - s) f_{i,j,t}(s)ds\right] \\
&\geq E\left[\frac{1}{2}\delta_\epsilon^2 1_{[|\delta_{i,j,t}(\alpha,\alpha^*)|>\delta_\epsilon]} + \frac{1}{2}\delta_{i,j,t}(\alpha,\alpha^*)^2 1_{[|\delta_{i,j,t}(\alpha,\alpha^*)|\leq\delta_\epsilon]}\right] \\
&\geq \frac{1}{2}\delta_\epsilon^2 E[1_{[|\delta_{i,j,t}(\alpha,\alpha^*)|>\delta_\epsilon]}].
\end{aligned}
$$

The first inequality above comes from the fact that Assumption 2(ii.a) implies that for any $\delta > 0$ sufficiently small, we have $f_{i,j,t}(s) > \delta$ for $|s| < \delta$. Thus,

$$
\begin{aligned}
\Delta(\alpha) &:= \sum_{i=1}^n \sum_{j=1}^p E(\Delta_{i,j,t}(\alpha)) \\
&\geq \frac{1}{2}\delta_\epsilon^2 \sum_{i=1}^n \sum_{j=1}^p E[1_{[|\delta_{i,j,t}(\alpha,\alpha^*)|>\delta_\epsilon]}] \\
&= \frac{1}{2}\delta_\epsilon^2 \sum_{i=1}^n \sum_{j=1}^p P[|\delta_{i,j,t}(\alpha,\alpha^*)| > \delta_\epsilon] \\
&\geq \frac{1}{2}\delta_\epsilon^2 \sum_{(i,j)\in\mathcal{I}} P[|\delta_{i,j,t}(\alpha,\alpha^*)| > \delta_\epsilon] \\
&\geq \frac{1}{2}\delta_\epsilon^2 P[\cup_{(i,j)\in\mathcal{I}}\{|\delta_{i,j,t}(\alpha,\alpha^*)| > \delta_\epsilon\}] > 0,
\end{aligned}
$$

where the final inequality follows from Assumption 4(i.b). As α is arbitrary, the result follows. ■

Next, we establish the asymptotic normality of $T^{1/2}(\hat{\alpha}_T - \alpha^*)$. We use a method originally proposed by Huber (1967) and later extended by Weiss (1991). We first sketch the method before providing formal conditions and the proof.

Huber's method applies to our estimator $\hat{\alpha}_T$, provided that $\hat{\alpha}_T$ satisfies the asymptotic first order conditions

$$
\begin{aligned}
&T^{-1}\sum_{t=1}^T \left\{\sum_{i=1}^n \sum_{j=1}^p \nabla q_{i,j,t}(\cdot,\hat{\alpha}_T)\psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\hat{\alpha}_T))\right\} \\
&\quad = o_p(T^{1/2}),
\end{aligned} \tag{14}
$$

where $\nabla q_{i,j,t}(\cdot,\alpha)$ is the $\ell \times 1$ gradient vector with elements $(\partial/\partial\alpha_s)q_{i,j,t}(\cdot,\alpha)$, $s = 1,\ldots,\ell$, and $\psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\hat{\alpha}_T))$ is a generalized residual. Our first task is thus to ensure that Eq. (14) holds.

Next, we define

$$
\lambda(\alpha) := \sum_{i=1}^n \sum_{j=1}^p E[\nabla q_{i,j,t}(\cdot,\alpha)\psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\alpha))].
$$

With $\lambda(\alpha)$ continuously differentiable at α^* interior to $\mathbb{A}$, we can apply the mean value theorem to obtain

$$
\lambda(\alpha) = \lambda(\alpha^*) + Q_0(\alpha - \alpha^*), \tag{15}
$$

where Q_0 is an $\ell \times \ell$ matrix with $(1 \times \ell)$ rows $Q_{0,s} = \nabla'\lambda(\bar{\alpha}_{(s)})$, where $\bar{\alpha}_{(s)}$ is a mean value (different for each s) lying on the segment connecting α and α^*, $s = 1,\ldots,\ell$. It is straightforward to show that the correct specification ensures that $\lambda(\alpha^*)$ is zero. We will also show that

$$
Q_0 = -Q^* + O(\|\alpha - \alpha^*\|), \tag{16}
$$

where $Q^* := \sum_{i=1}^n \sum_{j=1}^p E[f_{i,j,t}(0)\nabla q_{i,j,t}(\cdot,\alpha^*)\nabla' q_{i,j,t}(\cdot,\alpha^*)]$ with $f_{i,j,t}(0)$ representing the value at zero of the density $f_{i,j,t}$ of $\varepsilon_{i,j,t} := Y_{it} - q_{i,j,t}(\cdot,\alpha^*)$, conditional on $\mathcal{F}_{t-1}$. Combining Eqs. (15) and (16) and putting $\lambda(\alpha^*) = 0$, we obtain

$$
\lambda(\alpha) = -Q^*(\alpha - \alpha^*) + O(\|\alpha - \alpha^*\|^2). \tag{17}
$$

The next step is to show that

$$
T^{1/2}\lambda(\hat{\alpha}_T) + H_T = o_p(1), \tag{18}
$$

where $H_T := T^{-1/2}\sum_{t=1}^T \eta_t^*$, with $\eta_t^* := \sum_{i=1}^n \sum_{j=1}^p \nabla q_{i,j,t}(\cdot,\alpha^*)\psi_{\theta_{ij}}(\varepsilon_{i,j,t})$. Eqs. (17) and (18) then yield the following asymptotic representation of our estimator $\hat{\alpha}_T$:

$$
T^{1/2}(\hat{\alpha}_T - \alpha^*) = Q^{*-1}T^{-1/2}\sum_{t=1}^T \eta_t^* + o_p(1). \tag{19}
$$

As we impose conditions sufficient to ensure that $\{\eta_t^*, \mathcal{F}_t\}$ is a martingale difference sequence (MDS), a suitable central limit theorem (e.g., Theorem 5.24 in White, 2001) is applied to Eq. (19) to yield the desired asymptotic normality of $\hat{\alpha}_T$:

$$
T^{1/2}(\hat{\alpha}_T - \alpha^*) \xrightarrow{d} N(0, Q^{*-1}V^*Q^{*-1}), \tag{20}
$$

where $V^* := E(\eta_t^*\eta_t^{*\prime})$.

We now strengthen the conditions given in the text to ensure that each step of the above argument is valid.

Assumption 2(iii)(a) There exists a finite positive constant f_0 such that for each i and t, each $\omega \in \Omega$, and each $y \in \mathbb{R}$, $f_{it}(\omega, y) \leq f_0 < \infty$; (b) There exists a finite positive constant L_0 such that for each i and t, each $\omega \in \Omega$, and each $y_1, y_2 \in \mathbb{R}$, $|f_{it}(\omega,y_1) - f_{it}(\omega,y_2)| \leq L_0|y_1 - y_2|$.

Next we impose sufficient differentiability of q_t with respect to α.

Assumption 3(ii) For each t and each $\omega \in \Omega$, $q_t(\omega,\cdot)$ is continuously differentiable on $\mathbb{A}$; (iii) For each t and each $\omega \in \Omega$, $q_t(\omega,\cdot)$ is twice continuously differentiable on $\mathbb{A}$.

To exploit the mean value theorem, we require that α^* belongs to $int(\mathbb{A})$, the interior of $\mathbb{A}$.

Assumption 4(ii) $\alpha^* \in int(\mathbb{A})$.

Next, we place domination conditions on the derivatives of q_t.

Assumption 5(iii) We define

$$
D_{1,t} := \max_{i=1,\ldots,n}\max_{j=1,\ldots,p}\max_{s=1,\ldots,\ell}\sup_{\alpha\in\mathbb{A}}|(\partial/\partial\alpha_s)q_{i,j,t}(\cdot,\alpha)|.
$$

Then (a) $E(D_{1,t}) < \infty$; (b) $E(D_{1,t}^2) < \infty$;

(iv) Let us define

$$
\begin{aligned}
D_{2,t} := &\max_{i=1,\ldots,n}\max_{j=1,\ldots,p}\max_{s=1,\ldots,\ell}\max_{h=1,\ldots,\ell} \\
&\times \sup_{\alpha\in\mathbb{A}}|(\partial^2/\partial\alpha_s\partial\alpha_h)q_{i,j,t}(\cdot,\alpha)|.
\end{aligned}
$$

Then (a) $E(D_{2,t}) < \infty$; (b) $E(D_{2,t}^2) < \infty$.

Assumption 6. (i) $Q^* := \sum_{i=1}^n \sum_{j=1}^p E[f_{i,j,t}(0)\nabla q_{i,j,t}(\cdot,\alpha^*)\nabla' q_{i,j,t}(\cdot,\alpha^*)]$ is positive definite; (ii) $V^* := E(\eta_t^*\eta_t^{*\prime})$ is positive definite.

Assumptions 3(ii) and 5(iii.a) are additional assumptions that help to ensure that Eq. (14) holds. Further imposing Assumptions 2(iii), 3(iii.a), 4(ii), and 5(iv.a) suffices to ensure that Eq. (17) holds. The additional regularity provided by Assumptions 5(iii.b), (iv.b), and 6(i) ensures that Eq. (18) holds. Assumptions 5(iii.b) and 6(ii) help ensure the availability of the MDS central limit theorem. We now have conditions that are sufficient to prove the asymptotic normality of our MVMQ-CAViaR estimator.

Proof of Theorem 2. As outlined above, we first prove

$$T^{-1/2}\sum_{t=1}^{T}\left\{\sum_{i=1}^{n}\sum_{j=1}^{p}\nabla q_{i,j,t}(\cdot,\hat{\alpha}_T)\psi_{\theta_{ij}}(Y_{it}-q_{i,j,t}(\cdot,\hat{\alpha}_T))\right\} = o_p(1). \tag{21}$$

The existence of $\nabla q_{i,j,t}$ is ensured by Assumption 3(ii). Let e_i be the $\ell\times 1$ unit vector with the ith element equal to one and the rest zero, and let

$$G_s(c) := T^{-1/2}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p}\rho_{\theta_{ij}}(Y_{it}-q_{i,j,t}(\cdot,\hat{\alpha}_T+ce_s)),$$

for any real number c. Then, by the definition of $\hat{\alpha}_T$, $G_s(c)$ is minimized at $c=0$. Let $H_s(c)$ be the derivative of $G_s(c)$ with respect to c from the right. Then

$$H_s(c) = -T^{-1/2}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p}\nabla_s q_{i,j,t}(\cdot,\hat{\alpha}_T+ce_s)\psi_{\theta_{ij}} \times (Y_{it}-q_{i,j,t}(\cdot,\hat{\alpha}_T+ce_s)),$$

where $\nabla_s q_{i,j,t}(\cdot,\hat{\alpha}_T+ce_s)$ is the sth element of $\nabla q_{i,j,t}(\cdot,\hat{\alpha}_T+ce_s)$. Using the facts that (i) $H_s(c)$ is non-decreasing in c and (ii) for any $\epsilon>0$, $H_s(-\epsilon)\le 0$ and $H_s(\epsilon)\ge 0$, we have

$$\begin{aligned}|H_s(0)| &\le H_s(\epsilon)-H_s(-\epsilon)\\ &\le T^{-1/2}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p}|\nabla_s q_{i,j,t}(\cdot,\hat{\alpha}_T)|1_{[Y_{it}-q_{i,j,t}(\cdot,\hat{\alpha}_T)=0]}\\ &\le T^{-1/2}\max_{1\le t\le T}D_{1,t}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p}1_{[Y_{it}-q_{i,j,t}(\cdot,\hat{\alpha}_T)=0]},\end{aligned}$$

where the last inequality follows from the domination condition imposed in Assumption 5(iii.a). Because $D_{1,t}$ is stationary, $T^{-1/2}\max_{1\le t\le T}D_{1,t}=o_p(1)$. The second term is bounded in probability given Assumption 2(i), (ii.a) (see Koenker and Bassett, 1978, for details): that is,

$$\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p}1_{[Y_{it}-q_{i,j,t}(\cdot,\hat{\alpha}_T)=0]} = O_p(1).$$

Since $H_s(0)$ is the sth element of $T^{-1/2}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{p}\nabla q_{i,j,t}(\cdot,\hat{\alpha}_T)\,\psi_{\theta_{ij}}(Y_{it}-q_{i,j,t}(\cdot,\hat{\alpha}_T))$, the claim in (21) is proven.

Next, for each $\alpha\in\mathbb{A}$, Assumptions 3(ii) and 5(iii.a) ensure the existence and finiteness of the $\ell\times 1$ vector

$$\begin{aligned}\lambda(\alpha) &:= \sum_{i=1}^{n}\sum_{j=1}^{p}E[\nabla q_{i,j,t}(\cdot,\alpha)\psi_{\theta_{ij}}(Y_{it}-q_{i,j,t}(\cdot,\alpha))]\\ &= \sum_{i=1}^{n}\sum_{j=1}^{p}E\left[\nabla q_{i,j,t}(\cdot,\alpha)\int_{\delta_{i,j,t}(\alpha,\alpha^*)}^{0}f_{i,j,t}(s)ds\right],\end{aligned}$$

where $\delta_{i,j,t}(\alpha,\alpha^*) := q_{i,j,t}(\cdot,\alpha)-q_{i,j,t}(\cdot,\alpha^*)$ and $f_{i,j,t}(s) = (d/ds)F_{it}(s+q_{i,j,t}(\cdot,\alpha^*))$ represents the conditional density of $\varepsilon_{i,j,t} := Y_{it}-q_{i,j,t}(\cdot,\alpha^*)$ with respect to Lebesgue measure. The differentiability and domination conditions provided by Assumptions 3(iii) and 5(iv.a) ensure (e.g., by Bartle, 1966, Corollary 5.9) the continuous differentiability of $\lambda(\alpha)$ on $\mathbb{A}$, with

$$\nabla\lambda(\alpha) = \sum_{i=1}^{n}\sum_{j=1}^{p}E\left[\nabla\left\{\nabla' q_{i,j,t}(\cdot,\alpha)\int_{\delta_{i,j,t}(\alpha,\alpha^*)}^{0}f_{i,j,t}(s)ds\right\}\right].$$

Since α^* is interior to $\mathbb{A}$ by Assumption 4(ii), the mean value theorem applies to each element of $\lambda(\alpha)$ to yield

$$\lambda(\alpha) = \lambda(\alpha^*) + Q_0(\alpha-\alpha^*), \tag{22}$$

for α in a convex compact neighborhood of α^*, where Q_0 is an $\ell\times\ell$ matrix with $(1\times\ell)$ rows $Q_s(\bar{\alpha}_{(s)}) = \nabla'\lambda(\bar{\alpha}_{(s)})$, where $\bar{\alpha}_{(s)}$ is a mean value (different for each s) lying on the segment connecting α and α^* with $s=1,\ldots,\ell$. The chain rule and an application of the Leibniz rule to $\int_{\delta_{i,j,t}(\alpha,\alpha^*)}^{0}f_{i,j,t}(s)ds$ then give

$$Q_s(\alpha) = A_s(\alpha) - B_s(\alpha),$$

where

$$A_s(\alpha) := \sum_{i=1}^{n}\sum_{j=1}^{p}E\left[\nabla_s\nabla' q_{i,j,t}(\cdot,\alpha)\int_{\delta_{i,j,t}(\alpha,\alpha^*)}^{0}f_{i,j,t}(s)ds\right]$$

$$B_s(\alpha) := \sum_{i=1}^{n}\sum_{j=1}^{p}E[f_{i,j,t}(\delta_{i,j,t}(\alpha,\alpha^*))\nabla_s q_{i,j,t}(\cdot,\alpha)\nabla' q_{i,j,t}(\cdot,\alpha)].$$

Assumption 2(iii) and the other domination conditions (those of Assumption 5) then ensure that

$$\begin{aligned}A_s(\bar{\alpha}_{(s)}) &= O(\|\alpha-\alpha^*\|)\\ B_s(\bar{\alpha}_{(s)}) &= Q_s^* + O(\|\alpha-\alpha^*\|),\end{aligned}$$

where $Q_s^* := \sum_{i=1}^{n}\sum_{j=1}^{p}E[f_{i,j,t}(0)\nabla_s q_{i,j,t}(\cdot,\alpha^*)\nabla' q_{i,j,t}(\cdot,\alpha^*)]$. Letting $Q^* := \sum_{i=1}^{n}\sum_{j=1}^{p}E[f_{i,j,t}(0)\nabla q_{i,j,t}(\cdot,\alpha^*)\nabla' q_{i,j,t}(\cdot,\alpha^*)]$, we obtain

$$Q_0 = -Q^* + O(\|\alpha-\alpha^*\|). \tag{23}$$

Next, we have that $\lambda(\alpha^*)=0$. To show this, we write

$$\begin{aligned}\lambda(\alpha^*) &= \sum_{i=1}^{n}\sum_{j=1}^{p}E[\nabla q_{i,j,t}(\cdot,\alpha^*)\psi_{\theta_{ij}}(Y_{it}-q_{i,j,t}(\cdot,\alpha^*))]\\ &= \sum_{i=1}^{n}\sum_{j=1}^{p}E(E[\nabla q_{i,j,t}(\cdot,\alpha^*)\psi_{\theta_{ij}}(Y_{it}-q_{i,j,t}(\cdot,\alpha^*))\mid\mathcal{F}_{t-1}])\\ &= \sum_{i=1}^{n}\sum_{j=1}^{p}E(\nabla q_{i,j,t}(\cdot,\alpha^*)E[\psi_{\theta_{ij}}(Y_{it}-q_{i,j,t}(\cdot,\alpha^*))\mid\mathcal{F}_{t-1}])\\ &= \sum_{i=1}^{n}\sum_{j=1}^{p}E(\nabla q_{i,j,t}(\cdot,\alpha^*)E[\psi_{\theta_{ij}}(\varepsilon_{i,j,t})\mid\mathcal{F}_{t-1}])\\ &= 0,\end{aligned}$$

as $E[\psi_{\theta_{ij}}(\varepsilon_{i,j,t})\mid\mathcal{F}_{t-1}] = \theta_{ij} - E[1_{[Y_{it}\le q^*_{i,j,t}]}\mid\mathcal{F}_{t-1}] = 0$, by definition of $q^*_{i,j,t}$ for $i=1,\ldots,n$ and $j=1,\ldots,p$ (see Eq. (3)). Combining $\lambda(\alpha^*)=0$ with Eqs. (22) and (23), we obtain

$$\lambda(\alpha) = -Q^*(\alpha-\alpha^*) + O(\|\alpha-\alpha^*\|^2). \tag{24}$$

The next step is to show that

$$T^{1/2}\lambda(\hat{\alpha}_T) + H_T = o_p(1) \tag{25}$$

where $H_T := T^{-1/2}\sum_{t=1}^{T}\eta_t^*$, with $\eta_t^* := \eta_t(\alpha^*)$ and $\eta_t(\alpha) := \sum_{i=1}^{n}\sum_{j=1}^{p}\nabla q_{i,j,t}(\cdot,\alpha)\ \psi_{\theta_{ij}}(Y_{it}-q_{i,j,t}(\cdot,\alpha))$. Let $u_t(\alpha,d) := \sup_{\{\tau:\|\tau-\alpha\|\le d\}}\|\eta_t(\tau)-\eta_t(\alpha)\|$. By the results of Huber (1967) and Weiss (1991), to prove (25) it suffices to show the following: (i) there exist $a>0$ and $d_0>0$ such that $\|\lambda(\alpha)\|\ge a\|\alpha-\alpha^*\|$ for $\|\alpha-\alpha^*\|\le d_0$; (ii) there exist $b>0$, $d_0>0$, and $d\ge 0$ such that $E[u_t(\alpha,d)]\le bd$ for $\|\alpha-\alpha^*\|+d\le d_0$; and (iii) there exist $c>0$, $d_0>0$, and $d\ge 0$ such that $E[u_t(\alpha,d)^2]\le cd$ for $\|\alpha-\alpha^*\|+d\le d_0$.

The condition that Q^* is positive-definite in Assumption 6(i) is sufficient for (i). For (ii), we have that for the given (small) $d>0$

$$u_t(\alpha,d) \le \sup_{\{\tau:\|\tau-\alpha\|\le d\}}\sum_{i=1}^{n}\sum_{j=1}^{p}\|\nabla q_{i,j,t}(\cdot,\tau)\psi_{\theta_{ij}}(Y_{it}-q_{i,j,t}(\cdot,\tau))$$

$$
\begin{aligned}
&- \nabla q_{i,j,t}(\cdot,\alpha)\psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\alpha))\| \\
&\le \sum_{i=1}^{n}\sum_{j=1}^{p} \sup_{\{\tau:\|\tau-\alpha\|\le d\}} \|\psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\tau))\| \\
&\quad \times \sup_{\{\tau:\|\tau-\alpha\|\le d\}} \|\nabla q_{i,j,t}(\cdot,\tau) - \nabla q_{i,j,t}(\cdot,\alpha)\| \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{p} \sup_{\{\tau:\|\tau-\alpha\|\le d\}} \|\psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\alpha)) \\
&\quad - \psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\tau))\| \sup_{\{\tau:\|\tau-\alpha\|\le d\}} \|\nabla q_{i,j,t}(\cdot,\alpha)\| \\
&\le npD_{2,t}d + D_{1,t}\sum_{i=1}^{n}\sum_{j=1}^{p} 1_{[|Y_{it}-q_{i,j,t}(\cdot,\alpha)|<D_{1,t}d]}
\end{aligned}
$$

using the following: (i) $\|\psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\tau))\| \le 1$; (ii) $\|\psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\alpha)) - \psi_{\theta_{ij}}(Y_{it} - q_{i,j,t}(\cdot,\tau))\| \le 1_{[|Y_{it}-q_{i,j,t}(\cdot,\alpha)|<|q_{i,j,t}(\cdot,\tau)-q_{i,j,t}(\cdot,\alpha)|]}$; and (iii) the mean value theorem applied to $\nabla q_{i,j,t}(\cdot,\tau)$ and $q_{i,j,t}(\cdot,\alpha)$. Hence, we have

$$E[u_t(\alpha,d)] \le npC_0 d + 2npC_1 f_0 d$$

for some constants C_0 and C_1, given Assumptions 2(iii.a), 5(iii.a), and (iv.a). Hence, (ii) holds for $b = npC_0 + 2npC_1 f_0$ and $d_0 = 2d$. The last condition (iii) can be similarly verified by applying the c_r-inequality to the last equation above with $d < 1$ (so that $d^2 < d$) and using Assumptions 2(iii.a), 5(iii.b), and (iv.b). As a result, Eq. (25) is verified.

Combining Eqs. (24) and (25) yields

$$Q^* T^{1/2}(\hat{\alpha}_T - \alpha^*) = T^{-1/2}\sum_{t=1}^{T}\eta_t^* + o_p(1).$$

However, $\{\eta_t^*, \mathcal{F}_t\}$ is a stationary ergodic martingale difference sequence (MDS). In particular, η_t^* is measurable-$\mathcal{F}_t$, and we can show that

$$
\begin{aligned}
E(\eta_t^*|\mathcal{F}_{t-1}) &= E\left(\sum_{i=1}^{n}\sum_{j=1}^{p}\nabla q_{i,j,t}(\cdot,\alpha^*)\psi_{\theta_{ij}}(\varepsilon_{i,j,t}) \mid \mathcal{F}_{t-1}\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{p}\nabla q_{i,j,t}(\cdot,\alpha^*)E(\psi_{\theta_{ij}}(\varepsilon_{i,j,t}) \mid \mathcal{F}_{t-1}) \\
&= 0
\end{aligned}
$$

because $E[\psi_{\theta_{ij}}(\varepsilon_{i,j,t}) \mid \mathcal{F}_{t-1}] = 0$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, p$. Assumption 5(iii.b) ensures that $V^* := E(\eta_t^*\eta_t^{*\prime})$ is finite. The MDS central limit theorem (e.g., Theorem 5.24 of White, 2001) applies, provided V^* is positive definite (as ensured by Assumption 6(ii)) and that $T^{-1}\sum_{t=1}^{T}\eta_t^*\eta_t^{*\prime} = V^* + o_p(1)$, which is ensured by the ergodic theorem. The standard argument now gives

$$V^{*-1/2}Q^*T^{1/2}(\hat{\alpha}_T - \alpha^*) \xrightarrow{d} N(0, I),$$

which completes the proof. ■

To establish the consistency of $\hat{Q}_T$, we strengthen the domination condition on $\nabla q_{i,j,t}$ and impose conditions on $\{\hat{c}_T\}$.

Assumption 5(iii)(c) $E(D_{1,t}^3) < \infty$.

Assumption 7. $\{\hat{c}_T\}$ is a stochastic sequence and $\{c_T\}$ is a nonstochastic sequence such that (i) $\hat{c}_T/c_T \xrightarrow{p} 1$; (ii) $c_T = o(1)$; and (iii) $c_T^{-1} = o(T^{1/2})$.

Proof of Theorems 3 and 4. Theorems 3 and 4 can be proved by extending similar results in White et al. (2008). We do not report the proof to save space, but the complete proof of Theorems 3 and 4 can be found at the following website: http://web.yonsei.ac.kr/thkim/downloadable.html.

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